

# Two-Sided Matching Meets Fair Division

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## Abstract

We introduce a new model for two-sided matching which allows us to borrow popular fairness notions from the fair division literature such as envy-freeness up to one good and maximin share guarantee. In our model, each agent is matched to multiple agents on the other side over whom she has additive preferences. We demand fairness for each side separately, giving rise to notions such as double envy-freeness up to one match (DEF1) and double maximin share guarantee (DMMS). Our main result is that when both sides have identical preferences, the round robin algorithm with a carefully designed agent ordering achieves (a slight strengthening of) DEF1, but this is impossible when even one side has heterogeneous preferences.

## 1 Introduction

Consider a group of agents seeking to divide some number of indivisible goods amongst themselves. Each agent has a utility function describing the value that they have for every possible bundle of goods, and each agent may have a different utility function. This is a canonical resource allocation problem that arises in estate division, partnership dissolution, and charitable donations, to name just a few. A central goal is to find an allocation of the goods that is *fair*.

One desirable notion of fairness is *envy-freeness* [9], which requires that no agent prefer another agent's allocation of goods to her own. This is a compelling definition but, due to the discrete nature of the problem, cannot always be satisfied. Instead, we must consider relaxed versions, with one popular relaxation being *envy-freeness up to one good* (EF1) [14, 4], which requires that any pairwise envy can be eliminated by removing a single good from the envied agent's allocation. An allocation satisfying EF1 always exists for a broad class of agent utility functions [14].

While quite general, the resource allocation model fails to capture some allocation settings that we might be interested in. In particular, it does not allow for the possibility of *two-sided* preferences, in which agents have preferences over "goods," but also "goods" have preferences over agents. For instance, when college courses are allocated to students, it is reasonable to assume that students have preferences over the courses they take, and that teachers in charge of courses also have preferences over the students they accept (perhaps measured by prerequisites or GPA).<sup>1</sup> As another example, consider the problem of matching social services to vulnerable individuals,<sup>2</sup> where individuals have preferences over the services they receive, and service providers have preferences over the individuals they serve (perhaps based on demographics, location, or synergy with existing clients).

Motivated by these applications, we consider a two-sided resource allocation setting in which we have two groups of agents, where each agent has preferences over agents on the other side. Each agent must be "matched" to a subset of agents on the other side, subject to a maximum degree constraint. In the courses-to-students example, the maximum degree constraint reflects that each student has an upper bound on the number of courses they can take, and each course has an upper bound on its enrollment capacity. Our goal is to find a many-to-many matching that provides fairness

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<sup>1</sup>The problem of assigning students to courses has been studied before [4, 15, 5], but these papers typically only consider the preferences of the students.

<sup>2</sup><http://csse.utoronto.ca/social-needs-marketplace>

to both sets of agents simultaneously. Our model generalizes the standard resource allocation setting, in which the maximum degree constraint for each good is one (each good can be allocated to at most one agent), for each agent is simply the total number of goods (one agent can receive all the goods, in principle), and the goods are indifferent to which agent they are assigned to.

As we have already alluded to, our model bridges the gap between fair division and two-sided matching. In the two-sided matching literature, it is generally assumed that each agent has ordinal preferences over the other side, and a matching is sought that is in some sense *stable* to individual or group deviations. We retain the same basic structure of two-sided preferences, while incorporating fairness notions from fair division.

**Our results.** As a natural tradeoff between expressiveness and succinctness, we restrict our attention to additive preferences, in which the utility for being matched to a group of agents is equal to the sum of utilities for being matched to each agent in the group individually. While conceptually simple, additive preferences have led to a rich body of work in fair division. We focus primarily on the case in which all agents on the same side have the same degree constraint, and the total maximum degree on both sides is equal. In this case, it is reasonable to seek a *complete* matching, which saturates the degree constraints of all the agents on both sides.

We begin by considering *double envy-freeness up to one match* (DEF1), requiring that EF1 hold for both sets of agents simultaneously. We show that a carefully designed version of the classic round robin algorithm finds a complete matching that guarantees a slight strengthening of DEF1 when agents on both sides have identical ordinal preferences. However, when agents on even one side can have non-identical preferences, then a complete matching satisfying the aforementioned strengthening is not guaranteed to exist except in very special cases.

We also ask whether it is possible to find matchings that satisfy *double maximin share guarantee* (DMMS), i.e., the maximin share guarantee for both sets of agents simultaneously. Even when both sides have identical preferences, a complete DMMS matching may not exist, in contrast to the one-sided fair division setting in which an MMS allocation is guaranteed to exist when agents have identical preferences. In general, we show that approximate DMMS and approximate DEF1 are incompatible, although in the special case where the degree constraint is equal to two we can achieve exact versions of both simultaneously.

**Related work.** Most related to our work is that of Patro et al. [16], who draw on the resource allocation literature to guarantee fairness for both producers and consumers on a two-sided platform. However, in their model, producers are always indifferent between the customers; thus, only one side has interesting preferences. Other work [7, 22] has focused on guaranteeing fairness in two-sided platforms over time, rather than in a one-shot setting.

The theories of matching and fair division each have a rich history. Traditional work in matching theory has focused on one-to-one or many-to-one matchings, beginning with the seminal work of Gale and Shapley [11] and finding applications in areas such as school choice [2, 1, 12], kidney exchange [20], and the famous US resident-to-hospital match.<sup>3</sup> We note that EF1 as a condition becomes vacuous whenever a set of agents has a maximum degree constraint of one, so we focus instead on the more general case of many-to-many matchings. This case has also been well-explored in the matching literature [18, 21, 19, 8], although that literature focuses on stability notions, which have a very different flavor to our guarantees.

Our work draws extensively on notions from the fair division literature, particularly envy-freeness and its relaxations [9, 4, 14] and the maximin share guarantee [4]. Prior work has studied the satisfiability of these properties in the resource allocation setting [6, 17, 13] but, to our knowledge, no work has considered satisfying them on both sides of a market simultaneously.

## 2 Preliminaries

For  $n \in \mathbb{N}$ , define  $[n] = \{0, \dots, n - 1\}$ . There are two disjoint groups of agents, denoted  $N^\ell$  (“left”) and  $N^r$  (“right”), of sizes  $n^\ell$  and  $n^r$ , respectively. For simplicity of notation, we write  $N^\ell = [n^\ell]$  and  $N^r = [n^r]$ ; when referring to an agent by only its index, the group she belongs to will be clear from the context. We use indices  $i \in [n^\ell]$  and  $j \in [n^r]$

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<sup>3</sup><https://www.nrmp.org/>

to refer to agents on the left and right, respectively. We are given degree constraints  $d^\ell$  and  $d^r$  such that each  $i \in N^\ell$  and each  $j \in N^r$  can be matched to at most  $d^\ell$  and  $d^r$  agents on the opposite side, respectively.

A (many-to-many) *matching*  $M$  is represented as a binary  $n^\ell \times n^r$  matrix, where  $M(i, j) = 1$  if  $i \in N^\ell$  and  $j \in N^r$  are matched, and  $M(i, j) = 0$  otherwise. With slight abuse of notation, we denote  $M_i^\ell = \{j \in N^r : M(i, j) = 1\}$  and  $M_j^r = \{i \in N^\ell : M(i, j) = 1\}$  as the sets of agents on the opposite side that agents  $i \in N^\ell$  and  $j \in N^r$  are matched to, respectively. We say that  $M$  is *valid* if it respects the degree constraints, i.e., if  $|M_i^\ell| \leq d^\ell$  for each  $i \in N^\ell$  and  $|M_j^r| \leq d^r$  for each  $j \in N^r$ . Hereinafter, we omit the term *valid*, but will always refer to *valid matchings*. We say that  $M$  is *complete* if  $|M_i^\ell| = d^\ell$  for each  $i \in N^\ell$  and  $|M_j^r| = d^r$  for each  $j \in N^r$ . Most of our results are for the case where the maximum total degree is equal on both sides, i.e.,  $n^\ell \cdot d^\ell = n^r \cdot d^r$ , which allows us to find a complete matching.<sup>4</sup>

Each agent  $i \in N^\ell$  has a valuation function  $u_i^\ell : N^r \rightarrow \mathbb{R}_{\geq 0}$  and each agent  $j \in N^r$  has a valuation function  $u_j^r : N^\ell \rightarrow \mathbb{R}_{\geq 0}$ . When agents  $i \in N^\ell$  and  $j \in N^r$  are matched, they simultaneously receive utilities  $u_i^\ell(j)$  and  $u_j^r(i)$ , respectively. We assume that utilities are additive. Thus, with slight abuse of notation, the utilities to agents  $i \in N^\ell$  and  $j \in N^r$  under matching  $M$  are  $u_i^\ell(M_i^\ell) = \sum_{j \in M_i^\ell} u_i^\ell(j)$  and  $u_j^r(M_j^r) = \sum_{i \in M_j^r} u_j^r(i)$ , respectively.

Our main constructive results take only the agents' preference orders as input. For agent  $i \in N^\ell$  (resp.  $j \in N^r$ ), we denote by  $\sigma_i^\ell$  (resp.  $\sigma_j^r$ ) a linear order over  $N^r$  (resp.  $N^\ell$ ) which is consistent with the valuation function  $u_i^\ell$  (resp.  $u_j^r$ ), i.e.,  $j \succ_{\sigma_i^\ell} j'$  whenever  $u_i^\ell(j) > u_i^\ell(j')$  (resp.  $i \succ_{\sigma_j^r} i'$  whenever  $u_j^r(i) > u_j^r(i')$ ).<sup>5</sup>

Inspired by envy-freeness up to one good (EF1) from classical fair division [4, 14], we define the following fairness guarantee in our setting.

**Definition 1** (Double Envy-Freeness Up To  $c$  Matches (DEF $c$ )). We say that matching  $M$  is *envy-free up to  $c$  matches (EF $c$ )* over  $N^\ell$  if for each pair of agents  $i, i' \in N^\ell$ , there exists  $S^\ell \subseteq M_{i'}^\ell$  with  $|S^\ell| \leq c$  such that  $u_i^\ell(M_i^\ell) \geq u_i^\ell(M_{i'}^\ell \setminus S^\ell)$ . Similarly, we say that it is *EF $c$  over  $N^r$*  if, for each pair of agents  $j, j' \in N^r$ , there exists  $S^r \subseteq M_{j'}^r$  with  $|S^r| \leq c$  such that  $u_j^r(M_j^r) \geq u_j^r(M_{j'}^r \setminus S^r)$ . We say that  $M$  is *DEF $c$*  if it is *EF $c$*  over both  $N^\ell$  and  $N^r$ .

When an algorithm takes as input only the preference rankings, it must ensure that the matching it returns is *DEF $c$*  for all possible valuation functions which could have induced the rankings. It is easy to observe that this is equivalent to satisfying the following stronger guarantee which uses the stochastic dominance (SD) relation.<sup>6</sup> This is akin to the SD-EF1 strengthening of EF1 [10, 3].

**Definition 2** (SD Double Envy-Freeness Up To  $c$  Matches (SD-DEF $c$ )). We say that matching  $M$  is *SD-envy-free up to  $c$  matches (SD-EF $c$ )* over  $N^\ell$  if, for every  $t \in [n^r]$ ,

$$\sum_{p=0}^t M(i, \sigma_i^\ell(p)) \geq \sum_{p=0}^t M(i', \sigma_i^\ell(p)) - c, \forall i, i' \in N^\ell,$$

and is *SD-EF $c$*  over  $N^r$  if, for every  $t \in [n^\ell]$ ,

$$\sum_{p=0}^t M(\sigma_j^r(p), j) \geq \sum_{p=0}^t M(\sigma_j^r(p), j') - c, \forall j, j' \in N^r.$$

$M$  is called *SD-DEF $c$*  if it is *SD-EF $c$*  over both  $N^\ell$  and  $N^r$ .

Finally, we extend a different fairness notion from classical fair division called the maximin share guarantee (MMS).

**Definition 3** ( $\alpha$ -Double Maximin Share Guarantee ( $\alpha$ -DMMS)). Let  $\mathcal{M}$  denote the set of valid matchings. The *maximin share value* of agent  $i \in N^\ell$  is defined as

$$MMS_i^\ell = \max_{M \in \mathcal{M}} \min_{i' \in N^\ell} u_i^\ell(M_{i'}^\ell),$$

<sup>4</sup>Our main algorithmic results can be extended to handle a relaxation of this assumption, as we discuss in Section 3.1.

<sup>5</sup>Ties among agents with equal utility are broken arbitrarily.

<sup>6</sup>While the SD relation is typically used to compare randomized outcomes, it can be used to compare deterministic ones as well.

and the maximin share value of agent  $j \in N^r$  is defined as

$$MMS_j^r = \max_{M \in \mathcal{M}} \min_{j' \in N^r} u_j^r(M_{j'}^r).$$

Given  $\alpha \in [0, 1]$ , matching  $M$  is called  $\alpha$ -maximin share fair ( $\alpha$ -MMS) over  $N^\ell$  if  $u_i^\ell(M_i^\ell) \geq \alpha \cdot MMS_i^\ell$  for every  $i \in N^\ell$ , and  $\alpha$ -MMS over  $N^r$  if  $u_j^r(M_j^r) \geq \alpha \cdot MMS_j^r$  for every  $j \in N^r$ . It is called  $\alpha$ -DMMS if it is  $\alpha$ -MMS for both  $N^r$  and  $N^\ell$ . When  $\alpha = 1$ , we simply write DMMS instead of 1-DMMS.

### 3 Double Envy-Freeness Up To One Match

In this section, we focus on double envy-freeness up to one match, more specifically, its strengthening SD-DEF1. In Section 3.1, we show that when both groups of agents have identical ordinal preferences, an SD-DEF1 matching always exists and can be computed efficiently. In Section 3.2, we show that SD-DEF1 matchings are generally not guaranteed to exist when agents in at least one group have non-identical ordinal preferences, except under further restrictions.

#### 3.1 Identical Ordinal Preferences On Both Sides

In this subsection, we consider the special case where agents on both sides have identical ordinal preferences, i.e.,  $\sigma_i^\ell = \sigma_{i'}^\ell$  for all  $i, i' \in N^\ell$  and  $\sigma_j^r = \sigma_{j'}^r$  for all  $j, j' \in N^r$ . We simply denote with  $\sigma^\ell$  and  $\sigma^r$  the ordinal preferences of the agents in  $N^\ell$  and  $N^r$ , respectively. Further, without loss of generality, we assume that  $\sigma^\ell = 0 \succ \dots \succ n^\ell - 1$  and  $\sigma^r = 0 \succ \dots \succ n^r - 1$ . Our goal is to design an efficient algorithm for finding an SD-EF1 matching in this case.

We begin with the simplest case in which both sides have equal number of agents ( $n^\ell = n^r = n$ ), both sides have equal degree constraints ( $d^\ell = d^r = d$ ), and  $n$  and  $d$  are coprime. Later, we progressively reduce the general case to this simple case.

In this simple case, let us denote  $\sigma = \sigma^\ell = \sigma^r = 0 \succ \dots \succ n - 1$ . We want to find an SD-DEF1 matching under which each agent is matched to exactly  $d$  agents on the opposite side. A natural idea is to let agents on one side pick agents on the other side in a round-robin fashion. That is, we construct an ordering  $R$  over agents on one side, and these agents take turns according to  $R$  in a cyclic fashion with each agent, in her turn, making one match to her most preferred agent (i.e. lowest indexed agent) on the opposite side who has less than  $d$  matches so far. A standard argument from classical fair division shows that regardless of the ordering  $R$ , the resulting matching will be SD-EF1 over the side that does the picking. However, as the example below shows, not all orderings  $R$  lead to a matching that also satisfies SD-EF1 over the other side.

**Example 1.** Consider the case where  $n = 5$  and  $d = 2$ . Suppose the ordering is  $R = 0 \succ \dots \succ 4$ . Then, agent 0 on the right will be matched to agents 0 and 1 on the left, while agent 1 on the right will be matched to agents 2 and 3 on the left. Given that the preference ranking for agents on the right is also  $0 \succ \dots \succ 4$ , SD-EF1 is violated as agent 1 significantly envies agent 0 on the right side.

We now show that when  $R$  is carefully designed, SD-EF1 can also be satisfied over the other side, resulting in SD-DEF1. Algorithm 1 takes as input parameters  $a \in [n]$  and  $x \in \{d, n - d\}$ , and for any choices of these parameters, constructs an ordering  $R$  over the agents on (say) the left side. Algorithm 2 then uses this ordering to run the round-robin procedure while respecting the degree constraints. The next result shows that for any choices of the parameters, the resulting matching is SD-DEF1.

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**Algorithm 1** Round-Robin-Ordering( $n, a, x$ )

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- 1: **for**  $i \in [n]$  **do**
  - 2:      $p = i \cdot x \pmod{n}$
  - 3:      $R(p) = a + i \pmod{n}$
  - 4: **return**  $R$
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**Algorithm 2** Restricted-Round-Robin-Coprime( $n, d$ )

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- 1: Choose  $a \in \{0, \dots, n-1\}$  and  $x \in \{d, n-d\}$
  - 2:  $R = \text{Round-Robin-Ordering}(n, a, x)$   
// Round-robin with ordering  $R$  over agents on the left
  - 3:  $M(i, j) = 0, \forall i, j \in [n]$
  - 4: **for**  $j \in [n], t \in [d]$  **do**
  - 5:      $M(R(j \cdot d + t \pmod{n}), j) = 1$
  - 6: **return**  $M$
- 

**Theorem 1.** When  $n^\ell = n^r = n$  and  $d^\ell = d^r = d$  are coprime, and both groups of agents have identical ordinal preferences, Algorithm 2 efficiently computes a complete SD-DEF1 matching.

*Proof.* To avoid the  $(\text{mod } n)$  notation in this proof, we will treat integers as belonging to the ring  $\mathbb{Z}/n\mathbb{Z}$  of integers modulo  $n$ . Thus, addition and multiplication will be modulo  $n$ , and multiplicative inverses will also be modulo  $n$ . Note that  $x \in \{d, n-d\} = \{d, -d\}$  is coprime with  $n$ , so  $x^{-1}$  exists.

We first claim that the ordering  $R$  constructed in Algorithm 1 is a valid ordering over the agents in  $N^\ell$ . Notice that because  $x \in \{d, -d\}$  is coprime with  $n$ ,  $(i \cdot x)_{i \in [n]} = [n]$ . Thus, each index of  $R$  is set exactly once in the for loop.  $R$  can equivalently be represented as  $R(p) = a + px^{-1}$  for all  $p \in [n]$ . Because agents on the left take  $d$  turns in a cyclic fashion, it is convenient to think of an *extended* ordering  $R$  which is the original  $R$  concatenated with itself  $d$  times: interestingly, one can check that this still obeys  $R(p) = a + px^{-1}$  for all  $p \in [nd]$ .

Next, we argue that the matching returned is a valid complete matching. Notice that during the round-robin,  $d$  agents on the left that are consecutive in the ordering pick a given agent on the right before moving on to the next lowest indexed agent on the right. Further, each agent on the left gets  $d$  turns. Hence, it is easy to see that every agent is matched to exactly  $d$  agents on the opposite side.

As mentioned earlier, the fact that the returned matching  $M$  is SD-EF1 over  $N^\ell$  follows directly from the standard round-robin argument in classical fair division: given any pair of agents  $i, i' \in N^\ell$ , if we ignored the first turn taken by  $i'$ , then in each round agent  $i$  would get a turn before agent  $i'$  does, and hence, would not envy agent  $i'$  in the SD sense. It remains to show that  $M$  is also SD-EF1 over  $N^r$ . We show that for each agent  $j \in N^r$ , there exists an agent  $i \in N^\ell$  such that  $M_j^r = M_i^\ell$ . SD-EF1 over  $N^r$  will then follow from SD-EF1 over  $N^\ell$  given that  $\sigma^\ell = \sigma^r$ .

Let us focus on agent  $j \in N^r$ . Because agents on the right are picked from lowest-indexed to highest-indexed, agent  $j$  is picked by the  $d$  agents from  $N^\ell$  who appear consecutively in the (extended) ordering  $R$  at indices  $jd + t$  for  $t \in [d]$ . Given that  $R(p) = a + px^{-1}$  for all  $p \in [nd]$ , we immediately have

$$M_j^r = \{a + (jd + t)x^{-1} : t = [d]\}.$$

Next, let us focus on agent  $i \in N^\ell$ . If she is matched to some agent  $j \in N^r$  in a particular turn, then from the observation above, it must be that  $i = a + (jd + t)x^{-1}$  for some  $t \in [d]$ . Solving this for  $j$ , we get that  $j = ((i - a)x - t)d^{-1}$ . Varying  $t \in [d]$  in this equation gets us the  $d$  agents on the right that agent  $i$  is matched to:

$$M_i^\ell = \{((i - a)x - t)d^{-1} : t = [d]\}.$$

To show that for each  $j \in N^r$ , there exists  $i \in N^\ell$  with  $M_i^\ell = M_j^r$ , we take two cases.

If  $x = n - d = -d$ , then  $x^{-1} = (-d)^{-1} = -d^{-1}$ . In this case, it is easy to check that taking  $i = j$  suffices as

$$M_j^r = M_j^\ell = \{a - j - td^{-1} : t = [d]\}.$$

If  $x = d$ , then we have that

$$M_j^r = \{j + a + td^{-1} : t = [d]\},$$

while

$$M_i^\ell = \{i - a - td^{-1} : t = [d]\} = \{i - a - (d - 1 - t)d^{-1} : t = [d]\}.$$

Notice that  $M_j^r$  coincides with  $M_{j+2a-d^{-1}+1}^\ell$ . □

Algorithm 2 executes round-robin with the left side taking turns, and allows freely choosing  $a \in [n]$  and  $x \in \{d, n-d\}$  to decide their ordering. Note that if the right side takes turns instead, the algorithm still produces a complete SD-DEF1 matching. However, this extension does not find any new matchings. When  $x = n-d$ , the matching produced is symmetric ( $M_i^\ell = M_i^r$  for all  $i \in [n]$ ), and thus the same regardless of which side takes turns. When  $x = d$ , the allocations on one side are cyclic shift of the allocations on the other side. Hence, any matching produced by the right side taking turns can also be produced by the left side taking turns with appropriately chosen  $(a, x)$ .

What about allowing choices of  $x$  other than just  $d$  and  $n-d$ ? At least for  $n = 7$ ,  $d = 3$ , and  $a = 0$ , it is easy to check by hand that no other choices of  $x$  produce an SD-DEF1 matching. On the other end, could it be that some of the  $2n$  choices of  $(a, x)$  are redundant and lead to the same matching as other choices? The following result shows that in every instance, all  $2n$  choices lead to different matchings.

**Proposition 1.** For any inputs  $n$  and  $d$  to Algorithm 2, the  $2n$  possible choices of  $(a, x)$  result in distinct matchings.

*Proof.* Using the same reasoning as in the proof of Theorem 1, for each  $p \in [n]$ , we have

$$R(p) = a + p \cdot x^{-1} = \begin{cases} a + p \cdot d^{-1}, & \text{if } x = d, \\ a - p \cdot d^{-1} = a + n - p \cdot d^{-1}, & \text{if } x = n - d. \end{cases}$$

Note that if  $(a, x)$  generates an ordering  $(a, a_1, \dots, a_{n-1})$ , then  $(a, \hat{x})$  with  $\hat{x} \neq x$  generates the ordering  $(a, a_{n-1}, \dots, a_1)$ . Moreover, using different choices of  $a$  while fixing the choice of  $x$  generates orderings that are cyclically shifted versions of one another.

Let us denote with  $R$  and  $\hat{R}$  the orderings generated by Algorithm 1 using choices  $(a, x)$  and  $(\hat{a}, \hat{x})$ , respectively, and let  $M$  and  $\hat{M}$  denote the allocations returned by Algorithm 2 with orderings  $R$  and  $\hat{R}$ , respectively. Suppose for contradiction that  $(a, x) \neq (\hat{a}, \hat{x})$  but  $M = \hat{M}$ .

Let  $R = (a, a_1, \dots, a_{d-1}, \dots, a_{n-1})$ . Then, because agents in  $N^\ell$  pick agents on the opposite side in a round-robin fashion, we immediately have that  $M_0^r = \{a, a_1, \dots, a_{d-1}\}$ . If  $d < n/2$ , then  $a_{n-1} \notin M_1^r$ , while if  $d > n/2$ , then  $a \in M_1^r$ . Note that  $d = n/2$  is not possible because  $d$  and  $n$  are coprime. We now consider two cases.

**Case I:**  $x = \hat{x}$ . As  $R$  and  $\hat{R}$  are just cyclically shifted, it is easy to observe that  $M_0^r = \hat{M}_0^r$  implies  $a = \hat{a}$ , which is a contradiction.

**Case II:**  $x \neq \hat{x}$ . Let  $R'$  be the ordering obtained with  $(a, \hat{x})$ . Then,  $R' = (a, a_{n-1}, \dots, a_{d-1}, \dots, a_1)$ . Further, note that  $\hat{R}$  is a cyclically shifted version of  $R'$ , and we want  $\hat{M}_0^r = M_0^r = \{a, a_1, \dots, a_{d-1}\}$ . It is easy to notice that the only way to obtain this is by setting  $\hat{a} = a_{d-1}$ , which will induce  $\hat{R} = (a_{d-1}, \dots, a_1, a, a_{n-1}, \dots, a_d)$ , making  $a_{d-1}, \dots, a_1, a$  the first  $d$  agents. If  $d < n/2$ , then we have  $a_{n-1} \in \hat{M}_1^r$  but  $a_{n-1} \notin M_1^r$ , while if  $d > n/2$ , then we have  $a \notin \hat{M}_1^r$  but  $a \in M_1^r$ . In either case, we have that  $M_1^r \neq \hat{M}_1^r$ , which is a contradiction.  $\square$

While the  $2n$  choices of  $(a, x)$  lead to distinct complete SD-DEF1 matchings, they do not generate all possible complete SD-DEF1 matchings, as the following example shows.

**Example 2.** Consider the instance with  $n = 5$ ,  $d = 2$ , and identical ordinal preferences on both sides. Let us focus on the following complete matching  $M$ .

- $M_0^\ell = M_0^r = \{0, 2\}$
- $M_1^\ell = M_1^r = \{1, 3\}$
- $M_2^\ell = M_2^r = \{0, 4\}$
- $M_3^\ell = M_3^r = \{1, 4\}$
- $M_4^\ell = M_4^r = \{2, 3\}$

	$x = d$	$x = n - d$
$a = 0$	[0, 3, 1, 4, 2]	[0, 2, 4, 1, 3]
$a = 1$	[1, 4, 2, 0, 3]	[1, 3, 0, 2, 4]
$a = 2$	[2, 0, 3, 1, 4]	[2, 4, 1, 3, 0]
$a = 3$	[3, 1, 4, 2, 0]	[3, 0, 2, 4, 1]
$a = 4$	[4, 2, 0, 3, 1]	[4, 1, 3, 0, 2]

Table 1: Round-robin orderings returned by Algorithm 1 for different choices of  $(a, x)$  when  $n = 5$  and  $d = 2$ .

It is easy to verify that this matching is SD-DEF1. Without loss of generality, we can assume that agents in  $N^\ell$  choose agents on the opposite side during the round robin procedure. Then, agents 2 and 3 should be in the last two positions of the round-robin ordering as they are matched to the least preferred agent in  $N^r$ .

Table 1 shows the round robin orderings produced by all the choices of  $(a, x)$ . The reader can verify that neither places agents 2 and 3 in the last two positions. Hence,  $M$  is not returned by any of the choices.

This leaves open the question of characterizing the set of all complete SD-EF1 matchings.

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**Algorithm 3** Restricted-Round-Robin( $n, d$ )

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- 1:  $g = \gcd(n, d)$ ,  $n' = n/g$ , and  $d' = d/g$
  - 2: **for**  $k, m \in [g]$  **do**
  - 3:      $\hat{i} = \{n' \cdot k + i : i \in [n']\}$
  - 4:      $\hat{j} = \{n' \cdot m + j : j \in [n']\}$
  - 5:      $M(\hat{i}, \hat{j}) = \text{Restricted-Round-Robin-Coprime}(n', d')$
  - 6: **return**  $M$
- 

Next, we address the case where  $n$  and  $d$  are not coprime by reducing it to the coprime case. Letting  $g = \gcd(n, d)$ , we divide both sides into  $g$  sub-groups of  $n' = n/g$  agents each. Then, we run Algorithm 2 a total of  $g^2$  times to match agents from each sub-group on the left to  $d' = d/g$  agents from each sub-group on the right. This matches each agent with exactly  $d$  agents from the opposite side. Note that we allow each of the  $g^2$  calls to Algorithm 2 to use arbitrary choices of  $a$  and  $x$ . Nonetheless, we show that the resulting complete matching must be SD-EF1.

**Theorem 2.** *When  $n^\ell = n^r = n$ ,  $d^\ell = d^r = d$ , and both groups of agents have identical ordinal preferences, Algorithm 3 efficiently computes a complete SD-EF1 matching.*

*Proof.* For  $k, m \in [g]$ , define the sets  $B_k^\ell = \{n' \cdot k + i : i \in [n']\}$  and  $B_m^r = \{n' \cdot m + j : j \in [n']\}$  as used in Algorithm 3. Note that Algorithm 3 calls Algorithm 2 on each pair  $(B_k^\ell, B_m^r)$ , and as a result, each agent in  $B_k^\ell$  is matched to exactly  $d'$  agents in  $B_m^r$ , and vice-versa.

We want to show that the overall matching  $M$  produced by Algorithm 3 is SD-DEF1. Let us first show that it is SD-EF1 over  $N^\ell$ . Consider two arbitrary agents  $i, i' \in N^\ell$ . Given that their preference rankings are  $0 \succ \dots \succ n-1$ , to show that SD-EF1 holds for these two agents, we need to show that for all  $t \in [n]$ ,

$$\sum_{p=0}^t M(i, p) \geq \sum_{p=0}^t M(i', p) - 1.$$

Fix  $t \in [n]$ . Because  $\{B_m^r : m \in [g]\}$  forms a partition of  $N^r = [n]$ , there exists a unique  $m \in [g]$  such that  $t \in B_m^r$ . For each  $m' < m$ , each of agents  $i$  and  $i'$  are connected to exactly  $d'$  agents in  $B_{m'}^r$ . Hence,  $\sum_{p=0}^{n' \cdot m - 1} M(i, p) = \sum_{p=0}^{n' \cdot m - 1} M(i', p) = m \cdot d'$ . Thus, it remains to show that

$$\sum_{p \in B_m^r : p \leq t} M(i, p) \geq \sum_{p \in B_m^r : p \leq t} M(i', p) - 1. \quad (1)$$

Let  $k$  and  $k'$  be such that  $i \in B_k^\ell$  and  $i' \in B_{k'}^\ell$ . Let  $R$  and  $R'$  denote the round-robin orderings constructed in Line 2 of Algorithm 2 when called on  $(B_k^\ell, B_m^r)$  and  $(B_{k'}^\ell, B_m^r)$ , respectively. Let  $i''$  be the agent whose position in  $R$  matches the position of  $i'$  in  $R'$ . Then,  $i''$  is matched to the same set of agents from  $B_m^r$  during the call to Algorithm 2 on  $(B_k^\ell, B_m^r)$  as  $i'$  is matched to during the call to Algorithm 2 on  $(B_{k'}^\ell, B_m^r)$ . Because Algorithm 2 produces an SD-DEF1 matching, Equation (1) holds when  $i'$  is replaced by  $i''$ , and therefore, must also hold for  $i'$ .

Due to the exact same argument, matching  $M$  is also SD-EF1 over  $N^\ell$ , as desired.  $\square$

Finally, we turn our attention to the general case in which we drop the constraints  $n^\ell = n^r$  and  $d^\ell = d^r$ . As noted in Section 2, we still require  $n^\ell \cdot d^\ell = n^r \cdot d^r$ . The following result shows that a trick of adding dummy agents to the side with fewer agents, running Algorithm 3 (Restricted-Round-Robin) appropriately, and then removing the dummy agents works. The key is to show that the removal of dummy agents reduces the degrees of the agents on the opposite side exactly as intended and SD-DEF1 is preserved.

**Theorem 3.** *When  $n^\ell \cdot d^\ell = n^r \cdot d^r$ , and both groups of agents have identical ordinal preferences, a complete SD-DEF1 matching always exists and can be computed efficiently.*

*Proof.* Without loss of generality, assume that  $n^\ell \leq n^r$ , and hence,  $d^\ell \geq d^r$ . We add  $n^r - n^\ell$  dummy agents with indices  $n^\ell, \dots, n^r - 1$  to the left, so each side has exactly  $n^r$  agents. We extend the preferences of the agents on the right so that the dummy agents, indexed higher than the real agents, appear at the bottom of their preference rankings.

Now, we run the Restricted-Round-Robin algorithm (Algorithm 3) with inputs  $n^r$  and  $d^\ell$ . Let  $M$  denote the matching returned. Note that while each agent on the left has the intended degree  $d^\ell$ , each agent on the right has degree  $d^\ell$ , instead of  $d^r$ .

Finally, we remove the dummy agents from the left, which reduces the degrees of the agents on the right who were matched to them. Let  $\widehat{M} = M([n^\ell], [n^r])$  be the matching  $M$  restricted to the real agents. Our goal is to show that  $\widehat{M}$  is a complete SD-DEF1 matching for the original problem.

First, note that all agents on the left still have degree  $d^\ell$  under  $\widehat{M}$ . To show that all agents on the right now have degree  $d^r$  under  $\widehat{M}$ , we need to show that they are matched to an equal number ( $d^\ell - d^r$ ) of dummy agents under  $M$ . Suppose this is not the case. Then, there exist agents  $j, j' \in N^r$  such that  $j$  is matched to at least two more dummy agents than  $j'$  under  $M$ . It is easy to check that this violates SD-DEF1 of  $M$ , which is a contradiction because Algorithm 3 returns an SD-DEF1 matching. Thus, after removing the dummy agents, the degree of all agents on the right drop to precisely  $d^r$ . Hence,  $\widehat{M}$  is a complete matching.

To show that  $\widehat{M}$  is SD-DEF1, note that it is trivially SD-EF1 over the left because  $M$  is SD-EF1 over the left and allocations to the agents on the left do not change. It is also SD-EF1 over the right because  $M$  is SD-EF1 over the right and exactly  $d^\ell - d^r$  least-preferred agents are removed from the allocations of every agent on the right.  $\square$

We note that it is possible to extend our constructive result slightly beyond the case of  $n^\ell \cdot d^\ell = n^r \cdot d^r$ . Without loss of generality, assume that  $n^\ell \cdot d^\ell < n^r \cdot d^r$ . First, note that in this case, no matching is complete. We can still make the degree of each agent on the left equal to  $d^\ell$ , but the best we can hope for is that the degrees of agents on the right differ by at most 1, i.e., they are either  $\lfloor n^\ell \cdot d^\ell / n^r \rfloor$  or  $\lceil n^\ell \cdot d^\ell / n^r \rceil$ .<sup>7</sup> In this case, the trick outlined in Theorem 3 only works when the dummy agents are added to the left side, i.e., if  $n^\ell \leq n^r$ . We conjecture that such an SD-DEF1 matching always exists even when  $n^\ell > n^r$ , but leave it as an open question.

## 3.2 Identical Ordinal Preferences On One Side

In Section 3.1, we focused on the case where both groups of agents have identical ordinal preferences. In this section, we relax this assumption, and consider the case where at least one group has heterogeneous ordinal preferences. An immediate question is whether Theorem 3 can be extended to show that a complete SD-DEF1 matching still exists.

We begin by showing a positive result in a restrictive case.

**Theorem 4.** *When  $n^\ell = n^r = n$ ,  $d^\ell = d^r = 2$ , and at least one group of agents has identical ordinal preferences, a complete SD-DEF1 matching always exists and can be computed in polynomial time.*



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**Algorithm 4** Three-Phase-Round-Robin-I ( $n, d, \pi^r$ )
 

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- Phase 1:* .....
- 1: **for**  $j = 0, \dots, n/2 - 1$  **do**
  - 2:     Match agent  $j$  on the right to her most preferred agent on the left with no existing matches.
- Phase 2:* .....
- 3: **for**  $j = n/2, \dots, n - 1$  **do**
  - 4:     Match agent  $j$  on the right with her most preferred agent on the left which already has one match.
- Phase 3:* .....
- 5: **for**  $j = n - 1, \dots, n/2$  **do**
  - 6:     Let  $i'$  be the agent on the left that agent  $j$  on the right is already matched to.
  - 7:     Let  $j' \neq j$  be the other agent on the right that  $i'$  is matched to ( $j'$  must exist).
  - 8:     Match both  $j$  and  $j'$  to agent  $i'$ 's most preferred agent on the left who has no existing connection.
- 

The high-level idea is as follows. Algorithm 4 finds the desired complete SD-DEF1 matching when  $n$  is even; a more intricate algorithm for the case of odd  $n$  is presented as Algorithm 5. Intuitively, the algorithm works in three phases.

Let us divide  $N^r$  into two sets:  $B_1^r = \{0, \dots, n/2 - 1\}$  and  $B_2^r = \{n/2, \dots, n - 1\}$ . In the first phase, one-by-one we match agents in  $B_1^r$  to their most preferred agent on the left who has no prior matches. We do so in the best-to-worst order over agents in  $B_1^r$  according to  $\sigma^\ell$ . Let  $B^\ell$  be the set of agents in  $N^\ell$  who now have degree 1; note that  $|B^\ell| = n/2$ .

In the second phase, we repeat a similar process, except with agents in  $B_2^r$ , still in the best-to-worst order according to  $\sigma^\ell$ , and by matching them to their most preferred agent in  $B^\ell$ . Note that at the end of this phase, all agents on the right have degree 1, while agents in  $B^\ell$  have degree 2 and agents in  $N^\ell \setminus B^\ell$  have degree 0.

In the last phase, we again consider agents in  $B_2^r$ , but now in the *worst-to-best* order according to  $\sigma^\ell$ . Note that each such agent  $j$  is already matched to some agent  $i' \in B^\ell$ , and agent  $i'$  is also matched to some agent  $j'$  from  $B_1^r$ . We connect both  $j$  and  $j'$  to the most preferred agent of  $j$  from  $N^\ell \setminus B^\ell$ . Thus, in this phase, all agents on the right gain one additional degree, while the agents in  $N^\ell \setminus B^\ell$  gain two degrees each.

The proof of correctness of this algorithm presented in the following lemma. Then, we present the algorithm for the case of odd  $n$  (Algorithm 5), and prove its correctness as Lemma 2.

**Lemma 1.** *When  $d = 2$  and  $n$  is even, algorithm 4 returns a complete SD-DEF1 allocation.*

*Proof.* First, observe that any agent in  $B_1^r$ , and any agent in  $B_2^r$  have one matching in  $B^\ell$  and one matching in  $N^\ell \setminus B^\ell$ . Denote with  $y_j$  and  $z_j$  the matchings of agent  $j \in N^r$  in  $B^\ell$  and in  $N^\ell \setminus B^\ell$ , respectively. Moreover notice that for every agent  $j \in B_1^r$  there is an agent  $j' \in B_2^r$ , such that  $y_j = y_{j'}$  and  $z_j = z_{j'}$ , as in phase 3 when  $j'$  is matched to  $z_j$ , the same agent is also matched to  $j$  if  $y_j = y_{j'}$ .

We can easily verify that the matching is *SD - EF1* over  $N^\ell$ , as each agent in  $N^\ell$  has one matching in  $B_1^r$ , and one matching in  $B_2^r$ . Now, we prove that the matching is also *SD - EF1* with respect to the agents in  $N^r$ . We consider the three following cases.

**Case 1:**  $j, j' \in B_1^r$ . Without loss of generality, we assume that  $j < j'$ . Then,  $y_j \succ_{\sigma_j^r} y_{j'}$ , as  $j$  chooses before  $j'$  in phase 1. Thus,  $j$  cannot envy  $j'$  for more than one matchings. Moreover, as  $j$  has one matching in  $B^\ell$ , and one matching in  $N^\ell \setminus B^\ell$ , and as  $j'$  prefers  $y_{j'}$  to any agent in  $N^\ell \setminus B^\ell$  (otherwise she could have chosen an agent in  $N^\ell \setminus B^\ell$  as in phase 1 all of them have no matchings), we conclude that  $j'$  does not envy  $j$  for more than one matchings.

**Case 2:**  $j, j' \in B_2^r$ . Without loss of generality, we assume that  $j < j'$ . Then,  $y_j \succ_{\sigma_j^r} y_{j'}$ , as  $j$  appears before  $j'$  in phase 2. Thus,  $j$  cannot envy  $j'$  for more than one matchings. On the other hand,  $z_{j'} \succ_{\sigma_{j'}^r} z_j$ , as  $j'$  appears before  $j$  on phase 3. Hence  $j'$  does not envy  $j$  for more than one matchings.

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<sup>7</sup>In case that  $\frac{n^\ell \cdot d^\ell}{n^r}$  is an integer, we can set this to be  $d^r$  and achieve exactly equal degrees on the right side too.

**Case 3:**  $j \in B_1^r$  and  $j' \in B_2^r$ . If  $j$  and  $j'$  share the same matchings, then obviously they don't envy each other. Otherwise, we denote with  $\hat{j}$  the agent in  $B_2^r$  that has the same matchings with  $j$ , and with  $\hat{j}'$  the agent in  $B_1^r$  that has the same matchings with  $j'$ . Then, the theorem follows from cases 2 and 3, as the matching is SD-EF1 with respect to  $j$  and  $\hat{j}$ , and with respect to  $j'$  and  $\hat{j}'$ .  $\square$

---

**Algorithm 5** Three-Phase-Round-Robin-II ( $n, d, \pi^r$ )

---

*Phase 1:* .....

- 1: **for**  $j = 0, \dots, \lceil n/2 \rceil - 1$  **do**
- 2:     Match agent  $j$  on the right to her most preferred agent on the left with no existing matches.

*Phase 2:* .....

- 3: **for**  $j = \lceil n/2 \rceil, \dots, n - 2$  **do**
- 4:     Match agent  $j$  on the right with her most preferred agent on the left which already has one match.
- 5: Let  $i'$  be the agent on the left that agent  $\lceil n/2 \rceil - 1$  on the right is matched to.
- 6: **if**  $i'$  has only one match **then**
- 7:     Match agent  $n - 1$  on the right to agent  $i'$  on the left.
- 8: **else**
- 9:     Match agent  $n - 1$  on the right to her most preferred agent on the left with no existing matches.
- 10: Match agent  $\lceil n/2 \rceil - 1$  on the right with agent  $i'$  on the left with exactly one existing match (such an  $i'$  must exist)

*Phase 3:* .....

- 11: **for**  $j = n - 1, \dots, \lceil n/2 \rceil$  **do**
  - 12:     Let  $i'$  be the agent on the left that agent  $j$  on the right is already matched to.
  - 13:     Let  $j' \neq j$  be the other agent on the right whose only match is to agent  $i'$ .
  - 14:     Match both agents  $j$  and  $j'$  to agent  $i'$ 's most preferred agent on the left who has no existing matches.
  - 15: Match the agent on the right with one existing match, to the agent on the left with one existing match.
- 

**Lemma 2.** *When  $d = 2$  and  $n$  is odd, algorithm 5 returns a complete DEF1 allocation.*

*Proof.* Intuitively, the algorithm works in three phases. Let us divide  $N^r$  into two sets:  $B_1^r = \{0, \dots, \lceil n/2 \rceil - 1\}$ , and  $B_2^r = \{\lceil n/2 \rceil, \dots, n - 1\}$ . In the first phase, one-by-one we match agents in  $B_1^r$  to their most preferred agent on the left who has no prior matches. We do so in the best-to-worst order over agents in  $B_1^r$  according to  $\sigma^\ell$ . Let  $B^\ell$  be the set of agents in  $N^\ell$  who now have degree 1; note that  $|B^\ell| = \lceil n/2 \rceil$ .

In the second phase, we repeat a similar process, except with agents in  $B_2^r \setminus n - 1$ , still in the best-to-worst order according to  $\sigma^\ell$ , and by matching them to their most preferred agent in  $B^\ell$ . Then if  $i'$  on the left side, with whom  $\lceil n/2 \rceil - 1$  on the right is matched to, has only one matching,  $n - 1$  on the right side connects with her, otherwise  $n - 1$  is matched with her best choice in  $B^\ell$ . At this point,  $\lfloor n/2 \rfloor$  agents in  $B^\ell$  have degree 2, and only has degree 1, and hence  $\lceil n/2 \rceil - 1$  agent on the right is matched to agent in  $B^\ell$  with degree 1. So, at the end of this phase, all agents on the right have degree 1, except for  $\lceil n/2 \rceil - 1$  that has degree 2, while agents in  $B^\ell$  have degree 2 and agents in  $N^\ell \setminus B^\ell$  have degree 0.

In the last phase, we again consider agents in  $B_2^r$ , but now in the *worst-to-best* order according to  $\sigma^\ell$ . Note that each such agent  $j$  is already matched to some agent  $i' \in B^\ell$ , and agent  $i'$  is also matched to some agent  $j'$  from  $B_1^r$ . Note that all such  $j'$  from  $B_1^r \setminus \lceil n/2 \rceil - 1$  has degree 1. Hence, if  $j' \neq \lceil n/2 \rceil - 1$  both  $j$  and  $j'$  to the most preferred agent of  $j$  from  $N^\ell \setminus B^\ell$ , otherwise only  $j$  is matched to her. At the end, there is one  $j$  in  $B_1^r$ , and one agent  $i$  in  $N^\ell \setminus B^\ell$  that have degree one, and we matched them.

First, observe that any agent in  $B_1^r \setminus \lceil n/2 \rceil - 1$ , and any agent in  $B_2^r$  has one matching in  $B^\ell$  and one matching in  $N^\ell \setminus B^\ell$ , while  $\lceil n/2 \rceil - 1$  has two matchings in  $B^\ell$ . Denote with  $y_j$  and  $z_j$  the connections of agent  $j \in N^r \setminus \lceil n/2 \rceil - 1$  in  $B^\ell$  and in  $N^\ell \setminus B^\ell$ , respectively.

We denote with  $j^*$  the agent in  $B_2^r$  that share the same matching on the left with  $\lceil n/2 \rceil - 1$  on the right and with  $\widehat{j}^*$  the agent in  $B_1^r$  that share the same matching on the left with  $j^*$ . We see that for every agent  $j' \in B_2^r \setminus j^*$  there is an agent  $j \in B_2^r$ , such that  $y_j = y_{j'}$  and  $z_j = z_{j'}$ , as in the third phase when  $j'$  is matched to  $z_j$ , the same agent is also matched to  $j$  if  $y_j = y_{j'}$ .

We can easily verify that the matching is  $SD - EF1$  over  $N^\ell$ , as each agent in  $N^\ell$  either has one matching in  $B_1^r$ , and one matching in  $B_2^r$ , or has two matchings in  $B_1^r$  but one of them is with  $\lceil n/2 \rceil - 1$ .

Now, we prove that the allocation is also  $SD - EF1$  with respect to the agents in  $N^r$ . We consider the three following cases.

**Case 1:**  $j, j' \in B_1^r$ . This case is similar as Case 1 of the proof of lemma 1, so we omit the details.

**Case 2:**  $j, j' \in B_2^r$ . This case is similar as Case 2 of the proof of lemma 1, so we omit the details.

**Case 3:**  $j \in B_1^r$  and  $j' \in B_2^r$ . First we know  $j$  prefers  $y_j$  to any agent in  $N^\ell \setminus B^\ell$ , and as  $j'$  has only one matching in  $B^\ell$ , we conclude that  $j$  does not envy  $j'$  for more than one matchings.

We denote and with  $\widehat{j}$  the agent in  $B_1^r$  that has the same matchings with  $j' \in B_2^r \setminus j^*$ .

We assume that  $j \neq \{\lceil n/2 \rceil - 1, \widehat{j}^*\}$ . Then, we know from case 2 that  $j'$  does not envy  $\widehat{j}$  for more than one matching, and similar  $j$ . If  $j = \lceil n/2 \rceil$ , then either  $j'$  and  $j$  share the same matching in  $B^\ell$ , or  $j'$  before line 10 has chosen a better agent rather than the one that  $j$  is matched to in this line.

Next, if  $j = \widehat{j}^*$  and  $j' = j^*$ ,  $j$  and  $j'$  share one matching, and  $j'$  does not envy  $j$  for more than one matchings, while if  $j' \neq j^*$  we know that  $j'$  prefers  $y_{j'}$  to  $y_{\widehat{j}^*}$ , as  $y_{\widehat{j}^*}$  has only one connection before line 10.  $\square$

*Proof of Theorem 4.* The proof follows from Lemmas 1 and 2.  $\square$

Next, we show that when we relax the restrictions placed in Theorem 4, a complete SD-DEF1 matching may no longer be guaranteed to exist. First, we allow one group to have degree constraint greater than 3, while the other group still has degree constraint equal to 2.

**Theorem 5.** *Even when  $n^\ell \cdot d^\ell = n^r \cdot d^r$ ,  $d^r = 2$ , and at least one group of agents has identical ordinal preferences, a complete SD-DEF1 matching is not guaranteed to exist.*

*Proof.* Fix  $d^\ell \geq 3$  and  $d^r = 2$ . Choose  $n^\ell = 12d^\ell$  and  $n^r = 6(d^\ell)^2$ . Let all agents in  $N^\ell$  have identical preference ranking over agents in  $N^r$ , given by  $0 \succ \dots \succ n^r - 1$ . To define the ordinal preferences of agents in  $N^r$  over agents in  $N^\ell$ , let us partition the agents in  $N^r$  into  $d^\ell$  blocks:  $B_m^r = \{6d^\ell \cdot m, \dots, 6d^\ell \cdot (m+1) - 1\}$  for  $m \in [d^\ell]$ . We choose one preference ranking  $\rho_m$  for each block  $B_m^r$ , and let all agents in the block have this preference ranking. The first three rankings  $\rho_0, \rho_1$ , and  $\rho_2$  are shown below. The agents not shown in these rankings (marked “remaining agents”) can appear in an arbitrary order at the end. Rankings  $\rho_3, \dots, \rho_{d^\ell-1}$  can be completely arbitrary.

- $\rho_0 = 0 \succ \dots \succ 6d^\ell - 1 \succ \text{remaining agents}$
- $\rho_1 = 0 \succ 1 \succ \dots \succ 3d^\ell - 1 \succ 6d^\ell \succ \text{remaining agents}$
- $\rho_2 = 3d^\ell \succ \dots \succ 6d^\ell - 1 \succ 6d^\ell \succ \text{remaining agents}$

We claim that this instance does not admit a complete SD-DEF1 matching. Suppose for contradiction that it does.

We start by showing that in such a matching, each agent in  $N^\ell$  should be matched to exactly one agent in  $B_m^r$  for every  $m \in [d^\ell]$ . Suppose for contradiction that this is not true; let  $m$  be the smallest index for which it fails. Then, because the total degree of agents in  $B_m^r$  is  $6d^\ell \cdot d^r = n^\ell$ , there must exist agents  $i, i' \in N^\ell$  such that agent  $i$  is matched to no agent in  $B_m^r$  while agent  $i'$  is matched to at least two agents in  $B_m^r$ ; recall that both agents are matched to exactly one agent in  $B_{m'}^r$  for each  $m' < m$ . This violates the SD-EF1 condition with respect to agents  $i$  and  $i'$ .

Consider the set  $S_0 = [6d^\ell] \subset N^\ell$ . We claim that each agent in  $B_0^r$  must be matched to exactly one agent in  $S_0$ . This is because each agent in  $S_0$  is matched to exactly one agent in  $B_0^r$ ,  $|S_0| = |B_0^r| = 6d^\ell$ , and if two agents in  $S_0$  are matched to the same agent  $j \in B_0^r$ , then some other agent  $\widehat{j} \in B_0^r$  would not be matched to any agent from  $S_0$ , which would violate SD-EF1 with respect to agents  $j$  and  $\widehat{j}$ .

Next, observe that agent  $6d^\ell \in N^\ell$  is also matched to exactly one agent  $j \in B_0^r$ , who must be matched to exactly one agent  $i \in S_0$ . We consider two cases.

**Case 1:** Suppose  $i \in [3d^\ell]$ . In this case, we observe that because each agent in  $N^\ell$  is matched to exactly one agent from  $B_1^r$ , there must be an agent  $j' \in B_1^r$  who is not matched to any agent from  $S_1 = [3d^\ell] \cup \{6d^\ell\} \subset N^\ell$  (this is because  $|S_1| = 3d^\ell + 1 < 6d^\ell = |B_1^r|$ ). In contrast, agent  $j$  is matched to two agents from  $S_1$  — agents  $6d^\ell$  and  $i$ . Thus, SD-EF1 is violated with respect to agents  $j$  and  $j'$ , which is a contradiction.

**Case 2:** Suppose  $i \in [6d^\ell] \setminus [3d^\ell]$ . In this case, we observe that because each agent in  $N^\ell$  is matched to exactly one agent from  $B_2^r$ , there must be an agent  $j'' \in B_2^r$  who is not matched to any agent from  $S_2 = ([6d^\ell] \setminus [3d^\ell]) \cup \{6d^\ell\} \subset N^\ell$  (this is because  $|S_2| = 3d^\ell + 1 < 6d^\ell = |B_2^r|$ ). In contrast, agent  $j$  is matched to two agents from  $S_2$  — agents  $6d^\ell$  and  $i$ . Thus, SD-EF1 is violated with respect to agents  $j$  and  $j''$ , which is a contradiction.  $\square$

Next, we require both groups to be equal in size and their degree constraint, but allow the degree constraint to be greater than 2. The proof is in ??.

**Theorem 6.** *Even when  $n^\ell = n^r = n$ ,  $d^\ell = d^r = d \geq 3$ , and at least one group of agents has identical ordinal preferences, a complete SD-DEF1 matching is not guaranteed to exist.*

*Proof.* Let  $d \geq 3$  and  $n = 4d$ . Let all agents in  $N^\ell$  have identical preferences over agents in  $N^r$  given by  $0 \succ \dots \succ n$ . To define the preferences of agents in  $N^r$  over  $N^\ell$ , let us partition the agents in  $N^r$  into  $d$  blocks:  $B_m^r = \{4m, \dots, 4(m+1) - 1\}$  for  $m \in [d]$ . We define a preference ranking  $\rho_m$  for each block  $B_m^r$ , and let all agents in the block have this preference ranking. The first three rankings  $\rho_0, \rho_1$ , and  $\rho_2$  are shown below. Like in the proof of Theorem 4, the agents not shown in these rankings (marked “remaining agents”) can appear in an arbitrary order at the end. Rankings  $\rho_3, \dots, \rho_{d-1}$  can be completely arbitrary.

- $\rho_0 = 0 \succ 1 \succ 2 \succ 3 \succ \dots$
- $\rho_1 = 0 \succ 1 \succ 4 \succ 5 \succ \dots$
- $\rho_2 = 2 \succ 3 \succ 4 \succ 5 \succ \dots$

Once again, we claim that this instance does not admit any complete SD-DEF1 matching. Suppose for contradiction that it does.

Like in the proof of Theorem 4, we start by claiming that under such a matching, each agent in  $N^\ell$  must be matched to exactly one agent in  $B_m^r$  for every  $m \in [d]$ . Suppose for contradiction that this is not true; let  $m$  be the smallest index for which the statement fails. Then, because the total degree in  $B_m^r$  is  $4d = |N^\ell|$ , there must exist agents  $i, i' \in N^\ell$  such that  $i$  is matched to no agent from  $B_m^r$ ,  $i'$  is matched to at least two agents from  $B_m^r$ , and both  $i$  and  $i'$  are matched to exactly one agent from  $B_{m'}^r$  for each  $m' < m$ . This would violate SD-EF1 with respect to agents  $i$  and  $i'$ .

Notice that each agent in  $B_0^r$  must be matched to exactly one agent from  $S_0 = \{0, 1, 2, 3\} \subset N^\ell$ . If this is not true, then because each agent in  $S_0$  is matched to exactly one agent from  $B_0^r$  and  $|S_0| = |B_0^r| = 4$ , we must have agents  $j, j' \in B_0^r$  such that agent  $j$  is matched to at least two agents from  $S_0$  while agent  $j'$  is matched to none of them. This would violate SD-EF1 with respect to agents  $j$  and  $j'$ . By a similar reasoning, every agent in  $B_1^r$  must be matched to exactly one agent in  $S_1 = \{0, 1, 4, 5\} \subset N^\ell$ , and every agent in  $B_2^r$  must be matched to exactly one agent in  $S_2 = \{2, 3, 4, 5\} \subset N^\ell$ .

Consider agent  $j \in B_0^r$  that agent 4 from  $N^\ell$  is matched to. By the first observation above, there must be a unique agent  $i \in S_0$  who is also matched to agent  $j$ . We take two cases.

**Case 1:** Suppose  $i \in \{0, 1\}$ . Then, agent  $j$  is matched to two agents from  $\{0, 1, 4\} \subset S_1$  — agents  $i$  and 4. In contrast, the agent  $j' \in B_1^r$  who is matched to agent 5 from  $S_1$  is not matched to any agent from  $\{0, 1, 4\} \subset S_1$  by the second claim above. Hence, SD-EF1 is violated with respect to agents  $j$  and  $j'$ , which is a contradiction.

**Case 2:** Suppose  $i \in \{2, 3\}$ . Then, agent  $j$  is matched to two agents from  $\{2, 3, 4\} \subset S_2$  — agents  $i$  and 4. In contrast, the agent  $j' \in B_2^r$  who is matched to agent 5 from  $S_2$  is not matched to any agent from  $\{2, 3, 4\} \subset S_2$  by the third claim above. Hence, SD-EF1 is violated with respect to agents  $j$  and  $j'$ , which is a contradiction.  $\square$

## 4 Double Maximin Share Guarantee

In this section, we focus first on the existence of DMMS matchings, and second on the existence of matchings that are DMMS and SD-DEF1 concurrently.

We begin by considering the case where agents on both sides have identical preferences, i.e.,  $u_i^\ell(j) = u_i^\ell(j)$ , for any pair of agents  $i, i' \in N^\ell$ , and any  $j \in N^r$ , and similarly  $u_j^r(i) = u_j^r(i)$ , for any pair of agents  $j, j' \in N^r$  and any  $i \in N^\ell$ . We show the following negative result, which stands in contrast to the one-sided fair division setting in which an MMS allocation is guaranteed to exist when agents have identical preferences.

While the proof is intricate, the counterexample itself is simple; it has  $n = n^\ell = n^r = 7$  and  $d = d^\ell = d^r = 3$ ,  $u_i^\ell(j) = n - j - 1$  for all  $i \in N^\ell$  and  $j \in N^r$ , and  $u_j^r(i) = n - i - 1$  for all  $j \in N^r$  and  $i \in N^\ell$ . The proof proceeds by showing that, in every valid matching, some agent must receive utility less than or equal to 8, but the MMS value of all agents is 9, thus yielding an  $8/9$  approximation.

**Theorem 7.** *A 0.89-DMMS matching is not guaranteed to exist, even when agents on both sides have identical preferences.*

*Proof.* We denote by  $u^\ell$  and  $u^r$  the cardinal preferences of the agents in  $N^\ell$  and  $N^r$  respectively. As the utilities are the same across the agents in the same group, we can define  $MMS^\ell = MMS_i^\ell$  for all  $i \in [n]$ , and  $MMS^r = MMS_j^r$  for all  $j \in [n]$ .

As noted earlier, consider an instance with  $n = 7$ ,  $d = 3$ ,  $u^\ell(j) = n - j - 1$  for all  $j \in [n]$ , and  $u^r(i) = n - i - 1$  for all  $i \in [n]$ . Thus, for any complete matching,  $\sum_{i \in N^\ell} u^\ell(M_i^\ell) = \sum_{j \in N^r} u^r(M_j^r) = 63$ . This means that  $MMS^\ell = MMS^r \leq 9$ , because if all agents receive equal utility then they each get utility 9. Next, we construct a matching  $M$  such that  $u^\ell(M_i^\ell) = 9$  for all  $i \in [n]$ .

Without loss of generality, assume that  $M_0^\ell$ ,  $M_1^\ell$ , and  $M_2^\ell$  all contain agent 0. Then, we know that agents 1 and 2 cannot be contained in these bundles, because then they would have value larger than 9, implying that some other agent receives utility less than 9. Without loss of generality, we assume that bundles  $M_3^\ell$ ,  $M_4^\ell$ , and  $M_5^\ell$  contain agent 1. Now, we observe that  $M_6^\ell = \{2, 3, 4\}$ , as there is no other way to have  $u^\ell(M_6^\ell) = 9$ . As 0 and 2 can not belong to the same bundle (such a bundle would be valued at least 10), we may assume without loss of generality that agent 2 is contained in  $M_3^\ell$ , and  $M_4^\ell$ . Then, the constraint that  $u^\ell(M_3^\ell) = u^\ell(M_4^\ell) = 9$  dictates that  $M_3^\ell = M_4^\ell = \{1, 2, 6\}$ . With these bundles fixed, it is easy to check that the only  $M_5^\ell$  that yields  $u^\ell(M_5^\ell) = 9$  is  $M_5^\ell = \{1, 3, 5\}$ . Lastly, without loss of generality, we may assume that  $M_0^\ell = M_1^\ell = \{0, 4, 5\}$ , and  $M_2^\ell = \{0, 3, 6\}$ . Hence, we conclude that the following matching is the only one (subject to permutations of  $N^\ell$ ) that satisfies MMS for agents on the left.

- $M_0^\ell = M_1^\ell = \{0, 4, 5\}$
- $M_2^\ell = \{0, 3, 6\}$
- $M_3^\ell = M_4^\ell = \{1, 2, 6\}$
- $M_5^\ell = \{1, 3, 5\}$
- $M_6^\ell = \{2, 3, 4\}$

Now, consider agents  $0 \in N^r$  and  $4 \in N^r$ . Both are matched to agents  $0 \in N^\ell$  and  $1 \in N^\ell$ , but agent  $0 \in N^r$  is matched to agent  $2 \in N^\ell$  while agent  $4 \in N^r$  is matched to agent  $6 \in N^\ell$ . Therefore,  $u^r(M_0^r) \neq u^r(M_4^r)$  (and this difference persists regardless of permutations of  $N^\ell$ ). It is therefore not the case that every agent on the right receives utility 9; in particular, one agent receives utility 8 or less, producing the approximation ratio  $\alpha = 8/9 < 0.89$ .  $\square$

While a DMMS matching may not exist, even when preferences are identical, we can exploit the algorithms presented in Section 3 to obtain an approximation to DMMS.

**Theorem 8.** *When  $n^\ell \cdot d^\ell = n^r \cdot d^r$  and both groups of agents have identical utility functions, every complete SD-DEF1 matching  $M$  is also  $\frac{1}{d^\ell}$ -MMS over  $N^\ell$ , and  $\frac{1}{d^r}$ -MMS over  $N^r$ .*

*Proof.* Let  $M$  be a complete matching that is SD-EF1 over  $N^\ell$ . Let  $u^\ell$  denote the common utility function for agents in  $N^\ell$  which induces (without loss of generality) the preference ranking  $0 \succ \dots \succ n^r - 1$  over  $N^r$ . We show that  $M$  is also  $1/d^\ell$ -MMS over  $N^\ell$ . By symmetry, the MMS approximation for  $N^r$  also follows.

First, note that each agent in  $N^\ell$  must be matched to at least one agent in  $\{0, \dots, \lceil n^\ell/d^r \rceil - 1\} \subset N^r$ . This is because if some agent  $i \in N^\ell$  is matched to none of these agents, then some other agent  $i' \in N^\ell$  must be matched to at least two of them, which would violate SD-EF1 over  $N^\ell$ .

Let  $\widehat{M}$  be a matching that is MMS over  $N^\ell$ ; it is easy to see that under identical cardinal valuations, an MMS matching must exist. Because  $(\lceil n^\ell/d^r \rceil - 2) \cdot d^r \leq (n^\ell/d^r - 1) \cdot d^r < n^\ell$ , there must exist  $i \in N^\ell$  such that  $\widehat{M}_i^\ell \cap \{0, \dots, \lceil n^\ell/d^r \rceil - 2\} = \emptyset$ . In other words, if  $j^* = \operatorname{argmax}_{j \in \widehat{M}_i^\ell} u^\ell(j)$ , then  $j^* \geq \lceil n^\ell/d^r \rceil - 1$ . Notice that  $MMS^\ell \leq u^\ell(\widehat{M}_i^\ell) \leq d^\ell \cdot u^\ell(j^*)$  because  $|\widehat{M}_i^\ell| = d^\ell$ .

Now, because each agent in  $N^\ell$  is matched to at least one agent in  $\{0, \dots, \lceil n^\ell/d^r \rceil - 1\}$ , it follows that each agent receives utility at least  $u^\ell(j^*) \geq 1/d^\ell \cdot MMS^\ell$ , as desired.  $\square$

We next show an almost-matching upper bound that can be achieved by any SD-DEF1 matching, to complement Theorem 8. In fact, we show a more general result that trades off the approximation to DMMS with the approximation to double envy-freeness.

**Theorem 9.** *When  $n \geq d^2$ , a  $\frac{c+2}{d}$ -DMMS matching is incompatible with a SD-DEFc matching for all  $c \in [d]$ , even when agents on both sides have identical preferences.*

*Proof.* We present an example in which there is no matching that is concurrently  $\frac{c+2}{d}$ -MMS and SD-EF $c$  over  $N^\ell$ .

Let  $n$  be divisible by  $d$ . We consider an instance in which, for all  $i \in N^\ell$ ,  $u_i^\ell(j) = d + 1$  for  $j \in \{0, \dots, n/d - 2\}$ ,  $u_i^\ell(j) = 1$  for  $j \in \{n/d - 1, \dots, n/d + d - 2\}$ , and  $u_i^\ell(j) = 0$  for  $j \in \{n/d + d - 1, \dots, n\}$ . Hence,  $MMS^\ell$  is equal to  $d$ , as  $n - d$  agents of  $N^\ell$  can be connected with the first  $n/d - 1$  agents of  $N^r$  and acquire utility equal to  $d + 1$ , and the remaining  $d$  agents can have  $d$  connections in the set  $\{n/d - 1, \dots, n/d + d - 2\} \subset N^r$  and receive utility equal to  $d$ . Because there are less than  $n + d^2$  connections available in the subset  $\{0, \dots, n/d + d - 2\} \subset N^r$ , and  $n \geq d^2$ , it must be the case that there are agents on the left that have at most one connection to this subset of  $N^r$ . So, to achieve a SD-EF $c$  allocation with respect to the agents in  $N^\ell$ , each agent can have at most  $c + 1$  connections to this subset. But then only  $n - d$  out of  $n$  agents have utility at least  $d + 1$ , and the remaining ones have utility at most  $c + 1$ . So, there are agents that receive utility at most  $c + 1$  while  $MMS^\ell = d$ .  $\square$

For  $d \geq 3$ , Theorem 9 rules out the possibility of a matching that satisfies SD-DEF1 and DMMS. In the next theorem, we show that the two properties can be achieved simultaneously when  $d = 2$ .

**Theorem 10.** *When  $d = 2$ , a matching satisfying SD-DEF1 and DMMS exists and can be computed efficiently.*

*Proof.* Consider the following matching  $M$ . When  $n$  is even  $M_i^\ell = \{i, n - i - 1\}$  for  $i \in [d]$ , and when  $n$  is odd  $M_i^\ell = \{i, n - i - 1\}$  for  $i \in [d] \setminus \{\lfloor n/d \rfloor - 1, \lfloor n/d \rfloor, \lceil n/d \rceil\}$ , while  $M_{\lfloor n/d \rfloor - 1}^\ell = \{\lfloor n/d \rfloor - 1, \lfloor n/d \rfloor\}$ ,  $M_{\lfloor n/d \rfloor}^\ell = \{\lfloor n/d \rfloor - 1, \lceil n/d \rceil\}$  and  $M_{\lceil n/d \rceil}^\ell = \{\lfloor n/d \rfloor, \lceil n/d \rceil\}$ . Then,  $M_i^\ell = M_j^r$ , when  $i = j$ . It is easy to verify that this matching is SD-DEF1.

Let  $M_{min}^\ell = \operatorname{argmin}_{M_i^\ell \in M^\ell} u^\ell(M_i^\ell)$ . We show that  $u^\ell(M_{min}^\ell) = MMS^\ell$ . Assume for contradiction that there is another  $\widehat{M}^\ell$ , such that  $u^\ell(\widehat{M}_{min}^\ell) > u^\ell(M_{min}^\ell)$ , where  $\widehat{M}_{min}^\ell = \operatorname{argmin}_{\widehat{M}_i^\ell \in \widehat{M}^\ell} u^\ell(\widehat{M}_i^\ell)$ . Consider the cases that  $M_{min}^\ell = \{i, n - i + 1\}$ , with  $i \leq \lfloor n/d \rfloor - 2$  when  $n$  is odd and  $M_{min}^\ell = \{i, n - i + 1\}$ , with  $i \leq n/d - 1$ , when  $n$  is even. In order to have  $u^\ell(\widehat{M}_{min}^\ell) > u^\ell(M_{min}^\ell)$ , all the agents that are at least equal to  $n - i + 1$  should be included in a  $\widehat{M}_i^\ell$  with an agent smaller than  $i$ , as otherwise the minimum value is not increased. Notice that there are  $2 \cdot i$  connections that should be matched with  $2(i - 1)$  connections which is impossible. Next, if  $M_{min}^\ell = \{\lfloor n/d \rfloor - 1, \lfloor n/d \rfloor\}$ , or  $M_{min}^\ell = \{\lfloor n/d \rfloor - 1, \lceil n/d \rceil\}$ , then all the agents that are at least equal to  $\lceil n/d \rceil$  should be included in a  $\widehat{M}_i^\ell$  with an agent smaller than  $\lfloor n/d \rfloor - 1$  which is impossible. Lastly, if  $M_{min}^\ell = \{\lfloor n/d \rfloor, \lceil n/d \rceil\}$ , then all the agents that are at least equal to  $\lfloor n/d \rfloor$  should be included in a  $\widehat{M}_i^\ell$  with an agent at most equal to  $\lfloor n/d \rfloor - 1$  which is again impossible. Hence,  $u^\ell(M_{min}^\ell) = MMS^\ell$ .

With similar arguments it can be proved that  $u^r(M_{min}^r) = MMS^r$ , where  $M_{min}^r = \operatorname{argmin}_{M_j^r \in M^r} u^r(A_j^r)$   $\square$

Finally, we show that a strong impossibility persists even if we only require SD-EF1 on one side and MMS on the other. In the proof of Theorem 11, we solve an ILP to check nonexistence.

**Theorem 11.** *A matching that satisfies SD-EF1 over  $N^\ell$  and MMS over  $N^r$  is not guaranteed to exist, even when agents on both sides have identical preferences.*

*Proof.* We consider the instance that  $n^\ell = n^r = n = 11$ , and  $d^\ell = d^r = d = 3$ , while  $u^\ell = u_i^\ell(j) = n - j - 1$  for all  $i \in N^\ell$  and  $j \in N^r$  and  $u^r = u_j^r(i) = n - i - 1$  for all  $i \in N^\ell$  and  $j \in N^r$ . Finding  $MMS^\ell$  can be done using the following integer linear program  $ILP_1(n, d, v^\ell)$ :

$$\begin{aligned} & \text{maximize} && MMS^\ell \\ & \text{subject to} && \sum_{j \in [n]} M(i, j) \cdot v^\ell(j) \geq MMS^\ell, \quad \forall i \in [n] \\ & && \sum_{j \in [n]} M(i, j) = d, \quad \forall i \in [n] \\ & && \sum_{i \in [n]} M(i, j) = d, \quad \forall j \in [n] \\ & && M(i, j) \in \{0, 1\}, \quad \forall i, j \in [n] \end{aligned}$$

The first constraint ensures that the matchings is MMS over  $N^\ell$ , and the remaining ones ensure that the matching is valid.

Detecting whether it exists a matching that is SD-EF1 over  $N^r$  and MMS over  $N^\ell$  can be done using the following integer linear program  $ILP_2(n, d, v^\ell, MMS^\ell)$ :

$$\begin{aligned} & \sum_{i \in [r]} M(i, j) \geq \sum_{i \in [r]} M(i, j') - 1, \quad \forall j, j' \in [n], \forall r \in [n] \\ & \sum_{j \in [n]} M(i, j) \cdot v^\ell(j) \geq MMS^\ell, \quad \forall i \in [n] \\ & \sum_{j \in [n]} M(i, j) = d, \quad \forall i \in [n] \\ & \sum_{i \in [n]} M(i, j) = d, \quad \forall j \in [n] \\ & M(i, j) \in \{0, 1\}, \quad \forall i, j \in [n] \end{aligned}$$

The first constraint ensures that the matchings is SD-EF1 over  $N^r$ , the second one that it is MMS over  $N^\ell$ , and the remaining ones ensure that the matching is valid.

Using the above ILP programs, we check that the aforementioned instance does not admit a matching that is concurrently SD-EF1 over  $N^r$  and MMS over  $N^\ell$ .  $\square$

## 5 Discussion

We have introduced a model that bridges two-sided matching and fair division by requiring fairness on both sides of a matching market. We have shown that SD-EF1 can be achieved for agents on both sides, as long as all agents on the left (resp. right) share a common ordinal preference ranking over agents on the right (resp. left). When this condition is not satisfied, there may exist no matching that satisfies SD-DEF1. We have also shown that there may not exist a doubly MMS matching even when agents have identical preferences. While we do not rule out a good approximation to DMMS, we show that it is essentially impossible to obtain a good approximation to DMMS is one also requires SD-DEF1. In the appendix, we study other natural variants of DMMS, and the connection between DMMS and DEF1.

It is interesting to note that the proofs of Theorems 8 and 9 do not rely on the constraints that an agent in  $N^\ell$  can have up to  $d$  connections, and can be connected with an agent in  $N^r$  at most once. Therefore, these theorems also hold in a version of the one-sided fair division problem where there are  $n$  agents and  $n/d$  items with  $d$  copies each, and all the agents have identical preferences.

Many interesting avenues for future research remain. For example, future work could consider two-sided versions of other fairness notions. It would also be interesting to extend our results to allow for each agent to have a different degree constraint.

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## Appendix

### A Other Versions of DMMS

**DMMS without degree constraints** When we defined the maximin share of agent  $i$ , we only considered allocations  $A^\ell$  that respect the degree constraint. One may ask how much this constraint affects the maximin share, relative to an alternative definition where we maximize over all allocations in which each  $i \in N^\ell$  is matched with each  $j \in N^r$  at most once, but degree constraints are not required to hold.

Relative to this stronger version of MMS, a MMS matching over  $N^\ell$  that respects the degree constraint may only give a  $\Omega(1/n)$  approximation. Consider an example with  $n$  even and  $d = 2$ , and  $u^\ell(j) = n$  for any  $j \in [n/2 - 2]$ , while  $u^\ell(j) = 1$  for any remaining agents on the right. This means that  $n - 2$  agents on the left can be connected with one of the first agents on the right, and gain utility equal to  $n$ , while 2 agents can be connected with all the agents in  $\{n/2 - 1, \dots, n - 1\}$ , and gain utility equal to  $n/2 + 1$ . But as  $d = 2$ , this means that under a matching that respects the degree constraint, there is an agent that has utility equal to 2. Hence, the approximation is  $\Omega(1/n)$ .

**A weaker version of DMMS** We have seen that DMMS can not be achieved, even when agents have identical valuations,  $n^\ell = n^r = n$ , and  $d^\ell = d^r = d$ . However, consider the following relaxation of the MMS share (for simplicity, consider the MMS share of an agent  $i \in N^\ell$ ). Instead of maximizing over all matchings that satisfy the degree constraint, suppose that we instead maximize over all partitions  $\pi = \{\pi_1, \dots, \pi_k\}$  of  $N^r$  such that none of the resulting bundles contain more than  $d^\ell$  agents from  $N^r$ . That is, denoting by  $\Pi$  the space of all partitions of  $N^r$  that respect the degree constraint, the *weak MMS share* of agent  $i$  is defined as

$$wMMS_i^\ell = \max_{\pi \in \Pi} \min_{\pi_j \in \pi} u_i^\ell(\pi_j).$$

When all agents in  $N^\ell$  and  $N^r$  have identical valuations and  $n, d$  are common, weak MMS can always be achieved. First, find the partitions  $\pi^\ell$  and  $\pi^r$  that maximize the minimum value of any bundle, subject to respecting the degree constraint and breaking ties between partitions by selecting those that minimize the total number of bundles. Therefore,  $u_i(\pi_j^r) \geq wMMS_i^\ell$  for all  $i \in N^\ell$  and  $\pi_j^r \in \pi^r$ , and  $u_j(\pi_i) \geq wMMS_j^r$  for all  $j \in N^r$  and  $\pi_i^\ell \in \pi^\ell$ . Note that since each bundle formed by  $\pi^\ell$  and  $\pi^r$  can have at most  $d$  members, there must be exactly  $\lceil \frac{n}{d} \rceil$  bundles in each of  $\pi^\ell$  and  $\pi^r$  (if there were fewer bundles, one must contain more than  $d$  agents, and if there were more bundles then it would be possible to weakly increase the value of the lowest-valued bundle by merging two bundles). Now we can connect each agent in  $\pi_i^\ell$  to each agent in  $\pi_j^r$  for all  $i \in \{1, \dots, \lceil \frac{n}{d} \rceil\}$ . This matching respects degree constraints and satisfies weak MMS, by the definition of  $\pi^\ell$  and  $\pi^r$ .

### B DMMS & DEF1

In this section we explore the relationship between DMMS and DEF1.

**Proposition 2.** The existence of a DEF1 matching does not necessarily imply the existence of a DMMS matching.

*Proof.* We revisit the instance of the proof of theorem 7 and set  $y_0 = 0, y_3 = 1, y_2 = 2, y_5 = 3, y_6 = 4, y_1 = 5, y_4 = 6$ . It is easy to verify that the matching is DEF1, and the proposition follows.  $\square$

**Proposition 3.** A DMMS matching may not satisfy DFE1.

*Proof.* Consider the instance that  $n = 10, d = 3$ , and  $u^\ell(0) = u^\ell(1) = u^\ell(2) = 2$ , and  $u^\ell(j) = 1$  for any  $j \in \{3, \dots, 9\}$ , while  $u^r(i) = 1$  for any  $i \in [n]$ . Hence,  $MMS^\ell = 3$  and  $MMS^r = 3$ , and notice that any complete matching is DMMS. Now, consider the matching that  $M_0^\ell = M_1^\ell = M_2^\ell = \{0, 1, 2\}$ . Clearly,  $M$  is not DFE1.  $\square$

In the next theorem we prove that, given the existence of a DMMS matching that minimizes the number of agents that receive utility equal to  $MMS^\ell$  in  $N^\ell$  and simultaneously minimizes the number of agents that receive utility equal to  $MMS^r$  in  $N^r$ , we are guaranteed the existence of a DEF1 matching.

**Theorem 12.** A DMMS matching that minimizes the number of agents that receive utility equal to  $MMS^\ell$  in  $N^\ell$  and simultaneously minimizes the number of agents that receive utility equal to  $MMS^r$  in  $N^r$  is also DEF1.

*Proof.* We show that when a  $(d^\ell, d^r)$ -matching is MMS over  $N^\ell$  and minimizes the agents on the left that receive utility equal to  $MMS^\ell$ , it should also be EF1 it.

For contradiction, suppose that there is a  $(d^\ell, d^r)$ -matching which is MMS over  $N^\ell$ , minimizes the agents on the left that receive utility equal to  $MMS^\ell$ , but it is not EF1 over the left side. Let  $M_{i'}^\ell = \operatorname{argmin}_{M_i^\ell \in M^\ell} u^\ell(M_i^\ell)$  and  $i''$  be an agent such that EF1 is violated between  $i'$  and  $i''$ . Notice that  $u^\ell(M_{i'}^\ell) = MMS^\ell$ . As  $u^\ell(M_{i'}^\ell) < u^\ell(M_{i''}^\ell)$ , there is at least one pair  $j \in M_{i'}^\ell$  and  $j' \in M_{i''}^\ell$ , such that  $u^\ell(j) < u^\ell(j')$ . Now consider a different matching  $\widehat{M}^\ell$  such that  $\widehat{M}_i^\ell = M_i^\ell$ , for every  $i \neq \{i', i''\}$ ,  $\widehat{M}_{i'}^\ell = M_{i'}^\ell \setminus j \cup j'$ , and  $\widehat{M}_{i''}^\ell = M_{i''}^\ell \setminus j' \cup j$ . Then,  $u^\ell(\widehat{M}_{i'}^\ell) > u^\ell(M_{i'}^\ell)$ , as  $u^\ell(j) < u^\ell(j')$ . In addition, we have

$$u^\ell(\widehat{M}_{i''}^\ell) \geq u^\ell(\widehat{M}_{i''}^\ell \setminus j) > u^\ell(M_{i''}^\ell) = MMS^\ell$$

where the third inequality follows from the fact that the allocation is not EF1, as  $\widehat{M}_{i''}^\ell \setminus j = M_{i''}^\ell \setminus i'$ . Thus, we conclude in a matching in which the number of agents that receive utility  $MMS^\ell$  has been increased which is a contradiction.

The theorem follows by making similar arguments for  $N^r$ .  $\square$

One may wonder whether it is always possible, given a DMMS matching, to find a DMMS matching that satisfies the condition of Theorem 12. If this were true, then the existence of a DMMS matching would imply the existence of a DEF1 matching. However, we now show that simultaneously minimizing the number of agents that receive utility equal to  $MMS^\ell$  in  $N^\ell$  and the number of agents that receive utility equal to  $MMS^r$  in  $N^r$  may not be possible.

**Proposition 4.** There does not always exist a DMMS matching that minimizes the agents that receive utility equal to  $MMS^\ell$  in  $N^\ell$  and concurrently minimizes the agents that receive utility equal to  $MMS^r$  in  $N^r$ .

*Proof.* Consider the instance that  $n = 7, d = 3$ ,  $u^\ell(0) = u^\ell(1) = 3, u^\ell(2) = u^\ell(3) = u^\ell(4) = 1$ , and  $u^\ell(5) = u^\ell(6) = 0$ , while  $u^r(0) = 9, u^r(1) = u^r(2) = u^r(3) = u^r(4) = 3$ , and  $u^r(5) = u^r(6) = 0$ . Notice that  $MMS^r = 9$ , and it should exist three agents  $j, j',$  and  $j''$  such that  $M_j^r = M_{j'}^r = M_{j''}^r = \{0, 5, 6\}$ , and hence  $M_0^\ell = M_5^\ell = M_6^\ell = \{j, j', j''\}$ . Moreover, notice that  $MMS^\ell = 3$ , and the agents that receive utility equal to 3 are minimized if all the agents have two connections among the first 5 agents, except from one that is connected with 2, 3 and 4. Hence, we see that there is no way three agents on the left to be connected with the same three agents on the right, and the proposition follows. However, notice that the following matching:

- $M_0^r = \{0, 5, 6\}$
- $M_1^r = \{1, 2, 3\}$
- $M_2^r = \{4, 1, 2\}$
- $M_3^r = \{4, 1, 3\}$

- $M_4^T = \{4, 2, 3\}$
- $M_5^T = \{0, 5, 6\}$
- $M_6^T = \{0, 5, 6\}$

is DMMS and DFE1.

□