Fair and Efficient Social Decision-Making

CSCI 699

Voting: Axioms III

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Credit for the slides and most of visuals: Nisarg Shah and Ariel D. Procaccia

Randomized Voting Rules

- Input: preference profile
- Output: *distribution* over alternatives
  - To think about successful manipulations, we need numerical utilities
- u_i is consistent with $\succ_i$ if
$$a \succ_i b \Rightarrow u_i(a) \geq u_i(b)$$
- Strategyproofness:
  - For all i, $\overrightarrow{\succ}_{-i}$, $\succ_i$, $\succ_i'$, and u_i consistent with $\succ_i$

$$\mathbb{E}\left[u_i\left(f(\overrightarrow{\succ})\right)\right] \geq \mathbb{E}\left[u_i\left(f(\overrightarrow{\succ}_{-i}, \succ_i')\right)\right]$$

Randomized Voting Rules

- A (deterministic) voting rule is
  - unilateral if it only depends on one voter
  - duple if its range contains at most two alternatives

- A probability mixture f over rules $f_1, \dots, f_k$ is a rule given by some probability distribution $(\alpha_1, \dots, \alpha_k)$ s.t. on every profile $\overrightarrow{\succ}$, f returns $f_j(\overrightarrow{\succ})$ w.p. α_j.

- Example:
  - With probability 0.5, output the top alternative of a randomly chosen voter
  - With the remaining probability 0.5, output the winner of the pairwise election between a^* and b^*

- Theorem [Gibbard 77]
  - A randomized voting rule is strategyproof only if it is a probability mixture over unilaterals and duples.

Approximating Voting Rules

- Idea: Can we use strategyproof voting rules to approximate popular voting rules?
- Fix a rule (e.g., Borda) with a clear notion of score denoted $\mathrm{sc}(\vec{\succ}, a)$
- A randomized voting rule f is a c-approximation to sc if for every profile $\vec{\succ}$

$$\frac{\mathbb{E}[\mathrm{sc}\left(\vec{\succ}, f\left(\vec{\succ}\right)\right)}{\max_a \mathrm{sc}(\vec{\succ}, a)} \geq c$$

Approximating Borda

- Question: How well does choosing a random alternative approximate Borda?
  1. $\Theta(1/n)$
  2. $\Theta(1/m)$
  3. $\Theta(1/\sqrt{m})$
  4. $\Theta(1)$

- Theorem [Procaccia 10]:
  No strategyproof voting rule gives ${}^{1}/_{2} + \omega\left({}^{1}/_{\sqrt{m}}\right)$ approximation to Borda.

Quantitative GS Theorem

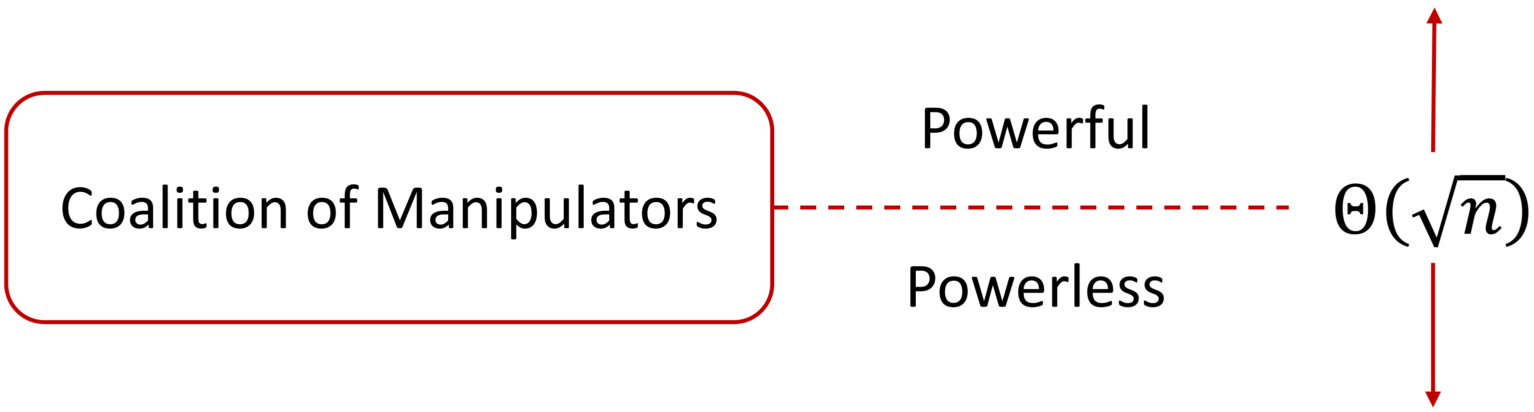
- Regarding the use of NP-hardness to circumvent GS
  - NP-hardness is hardness in the worst case
  - What happens in the average case?

- Theorem [Mossel-Racz '12]:
  - For every voting rule that is at least ϵ-far from being a dictatorship or having range of size 2…
  - …the probability that a uniformly random profile admits a manipulation is at least $p(n, m, 1/\epsilon)$ for some polynomial p

Coalitional Manipulations

- What if multiple voters collude to manipulate?
  - The following result applies to a wide family of voting rules called "generalized scoring rules".

- Theorem [Conitzer-Xia '08]:



Powerful = can manipulate with high probability

Interesting Tidbit

- Detecting a manipulable profile versus finding a beneficial manipulation

- Theorem [Hemaspaandra, Hemaspaandra, Menton '12]
  If integer factoring is NP-hard, then there exists a generalized scoring rule for which:
  - We can efficiently check if there exists a beneficial manipulation.
  - But finding such a manipulation is NP-hard.

Axiomatic Approach

Axiomatic Approach

- Axiom:
  - A requirement that the voting rule must behave in a certain way
- Goal:
  - Define a set of reasonable axioms, and search for voting rules that satisfy them together
  - Ultimate hope: a unique voting rule satisfies the set of axioms simultaneously!
  - What often happens: no voting rule satisfies the axioms together ☹

We have already seen axioms!

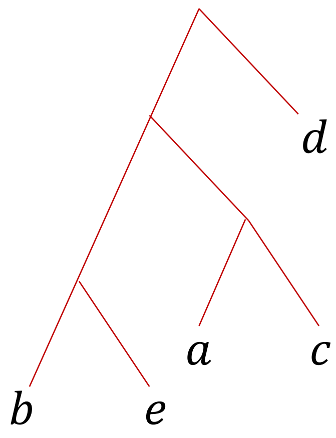
- Condorcet consistency
- Majority consistency
- Strategyproofness
- Ontoness
- Non-dictatorship
- Strong monotonicity
- Pareto optimality

Axiomatic Approach

- Some axioms are weak and satisfied by all natural rules
  - Unanimity:
    - If all voters have the same top choice, that alternative is the winner.

$$(top(\succ_i) = a \; \forall i \in N) \Rightarrow f(\vec{\succ}) = a$$

  - Q: How does this compare to Pareto optimality?
  - Pareto optimality is weak but still violated by natural voting methods like voting trees



Axiomatic Approach

- Anonymity:
  - Permuting the votes does not change the winner
  - In other words, voter identities don't matter
  - Example: these two profiles must have the same winner:
    {voter 1: $a \succ b \succ c$, voter 2: $b \succ c \succ a$}
    {voter 1: $b \succ c \succ a$, voter 2: $a \succ b \succ c$}
- Neutrality:
  - Permuting alternative names just permutes the winner accordingly
  - Example:
    - Say a wins on {voter 1: $a \succ b \succ c$, voter 2: $b \succ c \succ a$}
    - We permute all names: $a \rightarrow b$, $b \rightarrow c$, and $c \rightarrow a$
    - New profile: {voter 1: $b \succ c \succ a$, voter 2: $c \succ a \succ b$}
    - Then, the new winner must be b

Axiomatic Approach

- Neutrality is tricky for deterministic rules
  - Incompatible with anonymity
    - Consider the profile {voter 1: $a \succ b$, voter 2: $b \succ a$}
    - Without loss of generality, say a wins
    - Imagine a different profile: {voter 1: $b \succ a$, voter 2: $a \succ b$}
      - Neutrality ⇒ we exchanged $a \leftrightarrow b$, so winner must be b
      - Anonymity ⇒ we exchanged the votes, so winner must be a

- We usually only require neutrality for…
  - Randomized rules: E.g., a rule could satisfy both by choosing a and b as the winner with probability ½ each, on both profiles
  - Deterministic rules that return a set of tied winners: E.g., a rule could return $\{a, b\}$ as tied winners on both profiles.

Axiomatic Approach

- Consistency: If a is the winner on two profiles, it must be the winner on their union.

$$f(\overrightarrow{\succ}_1) = a \land f(\overrightarrow{\succ}_2) = a \Rightarrow f(\overrightarrow{\succ}_1 + \overrightarrow{\succ}_2) = a$$

  - Example: $\overrightarrow{\succ}_1 = \{a \succ b \succ c\}$, $\overrightarrow{\succ}_2 = \{a \succ c \succ b, b \succ c \succ a\}$
  - Then, $\overrightarrow{\succ}_1 + \overrightarrow{\succ}_2 = \{a \succ b \succ c, a \succ c \succ b, b \succ c \succ a\}$

- Theorem [Young '75]:
  - Subject to mild requirements, a voting rule is consistent if and only if it is a positional scoring rule!

Axiomatic Approach

- Weak monotonicity: If a is the winner, and a is "pushed up" in some votes, a remains the winner.
  - $f(\vec{\succ}) = a \Rightarrow f(\vec{\succ}') = a$, where
    - $b \succ_i c \Leftrightarrow b \succ_i' c, \forall i \in N,\ b, c \in A \backslash \{a\}$ (Order of others preserved)
    - $a \succ_i b \Rightarrow a \succ_i' b, \forall i \in N,\ b \in A \backslash \{a\}$ (a only improves)

- Contrast with strong monotonicity
  - SM requires $f(\vec{\succ}') = a$ even if $\vec{\succ}'$ only satisfies the 2nd condition
  - Too strong; only satisfied by dictatorial or non-onto rules [GS Theorem]

Axiomatic Approach

- Weak monotonicity is satisfied by most voting rules
  - Popular exceptions: STV, plurality with runoff

- But violation of weak monotonicity helps STV be hard to manipulate
  - Theorem [Conitzer-Sandholm ‘06]:
    “Every weakly monotonic voting rule is easy to manipulate on average.”

Axiomatic Approach

- STV violates weak monotonicity

7 voters	5 voters	2 voters	6 voters
a	b	b	c
b	c	c	a
c	a	a	b

7 voters	5 voters	2 voters	6 voters
a	b	a	c
b	c	b	a
c	a	c	b

- First c, then b eliminated
- Winner: a

- First b, then a eliminated
- Winner: c

Axiomatic Approach

- Arrow's Impossibility Theorem
  - Applies to social welfare functions (profile → ranking)
  - Independence of Irrelevant Alternatives (IIA): If the preferences of all voters between a and b are unchanged, the social preference between a and b should not change
  - Pareto optimality: If all prefer a to b, then the social preference should be $a \succ b$
  - Theorem: IIA + Pareto optimality ⇒ dictatorship

Axiomatic Approach

- Polynomial-time computability
  - Can be thought of as a desirable axiom
  - Two popular rules which attempt to make the pairwise comparison graph acyclic by inverting edges are NP-hard to compute:
    - Kemeny's rule: invert edges with minimum total weight
    - Slater's rule: invert minimum number of edges
  - Both rules can be implemented by straightforward integer linear programs
    - For small instances (say, up to 20 alternatives), NP-hardness isn't a practical concern.

Axiomatic Approach

Criterion / Method	Majority	Maj. loser	Mutual maj.	Condorcet	Cond. loser	Smith/ ISDA	LIIA	IIA	Cloneproof	Monotone	Consistency	Participation	Reversal symmetry	Polytime/ resolvable		Summable	Later-no- Harm	Later-no- Help	No favorite betrayal	Ballot type	Ranks =	Ranks >2
Approval	Rated[a]	No	No	No[b][c]	No	No[b]	Yes	Yes[d]	Yes[e]	Yes	Yes	Yes	Yes	O(N)	Yes	O(N)	No	Yes[f]	Yes	Approvals	Yes	No
Borda count	No	Yes	No	No[b]	Yes	No	No	No	Teams	Yes	Yes	Yes	Yes	O(N)	Yes	O(N)	No	Yes	No	Ranking	No	Yes
Bucklin	Yes	Yes	Yes	No	No	No	No	No	No	Yes	No	No	No	O(N)	Yes	O(N)	No	Yes	If equal preferences	Ranking	Yes	Yes
Copeland	Yes	Yes	Yes	Yes	Yes	Yes	No	No[b]	Teams, crowds	Yes	No[b]	No[b]	Yes	$O(N^2)$	No	$O(N^2)$	No[b]	No	No[b]	Ranking	Yes	Yes
IRV (AV)	Yes	Yes	Yes	No[b]	Yes	No[b]	No	No	Yes	No	No	No	No	$O(N^2)$	Yes[g]	O(N!)[h]	Yes	Yes	No	Ranking	No	Yes
Kemeny–Young	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No[b]	Spoilers	Yes	No[b][i]	No[b]	Yes	O(N!)	Yes	$O(N^2)$[j]	No[b]	No	No[b]	Ranking	Yes	Yes
Majority judgment[k]	Rated[l]	Yes[m]	No[n]	No[b][c]	No	No[b]	Yes	Yes[d]	Yes	Yes	No[o]	No[p]	Depends[q]	O(N)	Yes	O(N)[r]	No[s]	Yes	Yes	Scores[t]	Yes	Yes
Minimax	Yes	No	No	Yes[u]	No	No	No	No[b]	Spoilers	Yes	No[b]	No[b]	No	$O(N^2)$	Yes	$O(N^2)$	No[b][u]	No	No[b]	Ranking	Yes	Yes
Plurality/FPTP	Yes	No	No	No[b]	No	No[b]	No	No	Spoilers	Yes	Yes	Yes	No	O(N)	Yes	O(N)	N/A[v]	N/A[v]	No	Single mark	N/A	No
Score voting	No	No	No	No[b][c]	No	No[b]	Yes	Yes[d]	Yes	Yes	Yes	Yes	Yes	O(N)	Yes	O(N)	No	Yes	Yes	Scores	Yes	Yes
Ranked pairs	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No[b]	Yes	Yes	No[b]	No[p][b]	Yes	$O(N^3)$	Yes	$O(N^2)$	No[b]	No	No[p][b]	Ranking	Yes	Yes
Runoff voting	Yes	Yes	No	No[b]	Yes	No[b]	No	No	Spoilers	No	No	No	No	O(N)[w]	Yes	O(N)[w]	Yes	Yes[x]	No	Single mark	N/A	No[y]
Schulze	Yes	Yes	Yes	Yes	Yes	Yes	No	No[b]	Yes	Yes	No[b]	No[p][b]	Yes	$O(N^3)$	Yes	$O(N^2)$	No[b]	No	No[p][b]	Ranking	Yes	Yes
STAR voting	No[z]	Yes	No[aa]	No[b][c]	Yes	No[b]	No	No	No	Yes	No	No	Depends[ab]	O(N)	Yes	$O(N^2)$	No	No	No[ac]	Scores	Yes	Yes
Sortition, arbitrary winner[ad]	No	No	No	No[b]	No	No[b]	Yes	Yes	No	Yes	Yes	Yes	Yes	O(1)	No	O(1)	Yes	Yes	Yes	None	N/A	N/A
Random ballot[ae]	No	No	No	No[b]	No	No[b]	Yes	Yes	Yes	Yes	Yes	Yes	Yes	O(N)	No	O(N)	Yes	Yes	Yes	Single mark	N/A	No