Fair and Efficient Social Decision-Making

CSCI 699

Fair Division: Indivisible Goods

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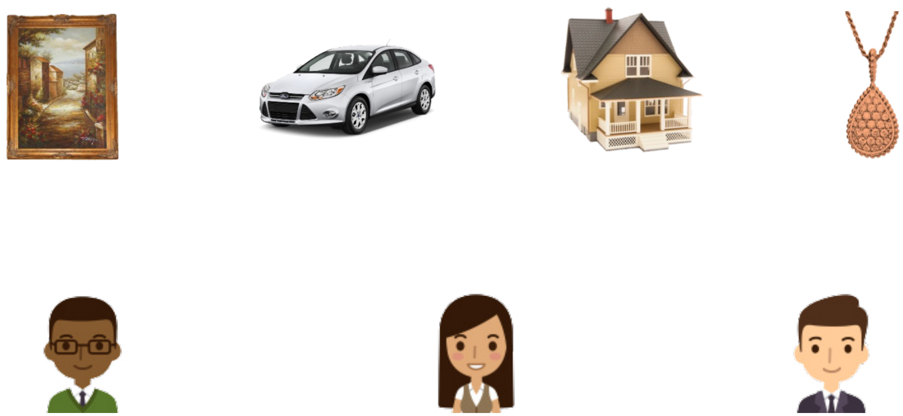
Credit for the slides : Nisarg Shah

Indivisible Goods

- Goods which cannot be shared among players



Model

- Agents: $N = \{1, 2, \ldots, n\}$

- Resource: Set of indivisible goods $M = \{g_1, g_2, \ldots, g_m\}$

- Allocation $A = (A_1, \ldots, A_n) \in \Pi_n(M')$ is a partition of M' for some $M' \subseteq M$.

- Each agent i has a valuation $v_i : M \to \mathbb{R}_+$
  - $v_i : M \to \mathbb{R}_-$ in the case of bads, $v_i : M \to \mathbb{R}$ for both goods and bads

- Additive Valuations: $\forall X \subseteq M, v_i(X) = \sum_{g \in X} v_i(g)$

Indivisible Goods

				
	8	7	20	5
	9	11	12	8
	9	10	18	3

Indivisible Goods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Indivisible Goods

				
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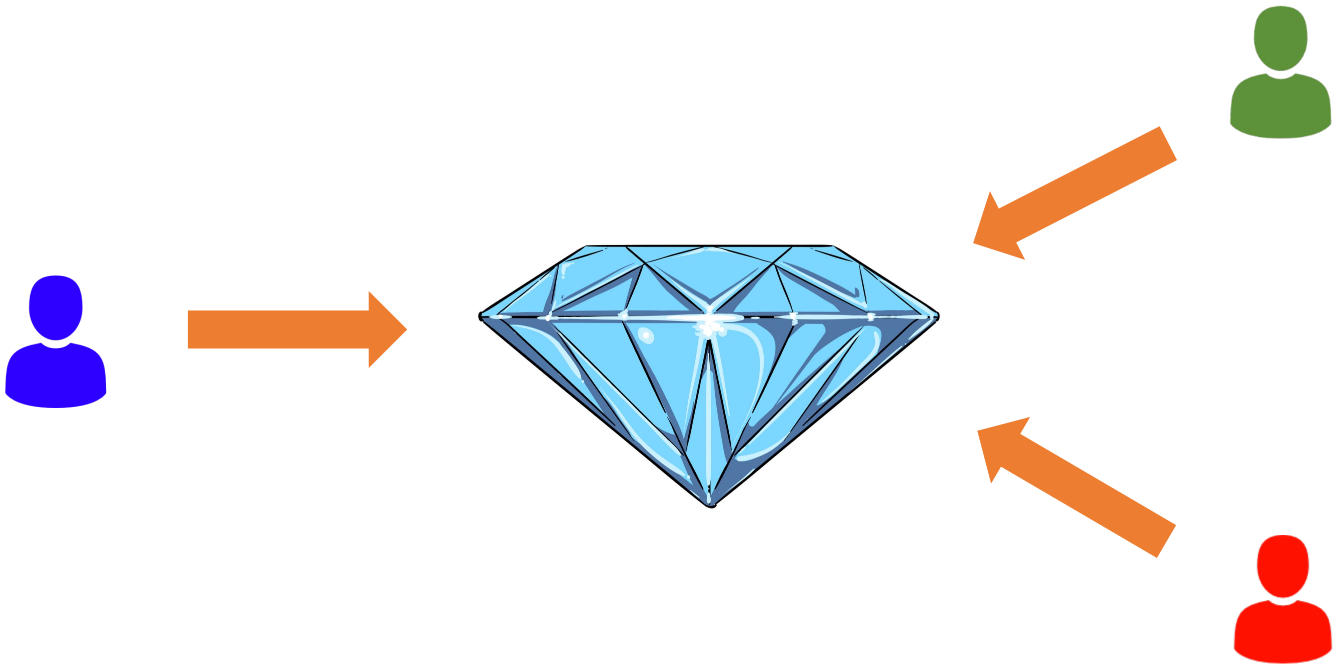
Indivisible Goods

				
	8	7	**20**	5
	9	11	12	**8**
	**9**	**10**	18	3

Need new guarantees!



Envy-Freeness up to One Good

Envy-Freeness up to One Good (EF1)

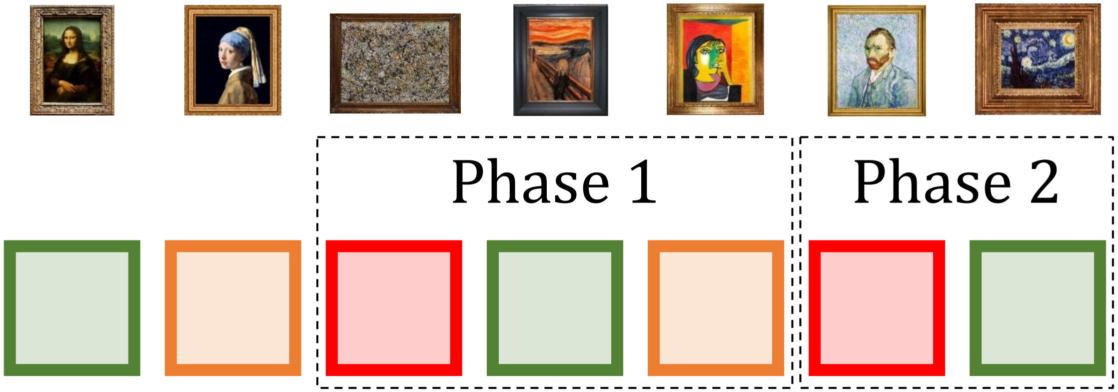
- An allocation is envy-free up to one good (EF1) if, for all agents i, j, there exists a good $g \in A_j$ for which

$$v_i(A_i) \geq v_i(A_j \setminus \quad)$$

- "Agent i may envy agent j, but the envy can be eliminated by removing a single good from j's bundle."

Round Robin Algorithm

- Fix an ordering of the agents σ.
- In round $k \bmod n$, agent σ_k selects their most preferred remaining good.
- Theorem: Round robin satisfies EF1.



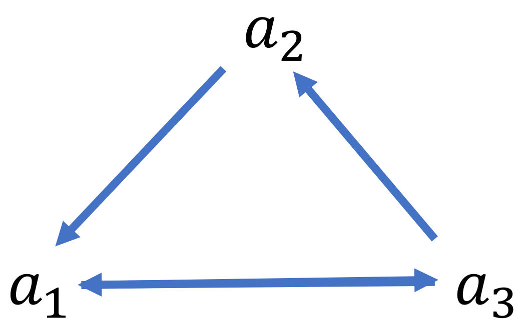
Animation Credit: Ariel Procaccia

Envy Cycle Elimination Algorithm

- Envy graph: Edge from i to j if i envies j

- Envy Cycle Elimination[Lipton et al. 2004]
  - One at a time, allocate a good to an agent that no one envies
  - While there is an envy cycle, rotate the bundles along the cycle.
    - Can prove this loop terminates in a polynomial number of steps

Envy Cycle Elimination Algorithm



	g_1	g_2	g_3	g_4	g_5	g_6
a_1	2	1	3	0	1	2
a_2	10	1	1	1	2	5
a_3	3	1	3	0	5	2

Envy Cycle Elimination Algorithm

- Theorem [Lipton et al. 2004]: Envy Cycle Elimination satisfies EF1
- Proof:
  - By induction on the number of rounds; let A^t be the allocation at the end of round t
  - For $t = 0$, A^0 is obviously EF1
  - Suppose that A^t is EF1
  - Case 1: At round $t + 1$, one more item is allocated
    - The item is allocated to a non-envied agent and therefore A_{t+1} is EF1
  - Case 2: At round $t + 1$, a cycle is eliminated

1. For each $i \in N$, $v_i(A_i^{t+1}) \geq v_i(A_i^t)$
2. For each $j \in N$, $\exists\ j' \in N$ such that $A_j^{t+1} = A_{j'}^t$
3. For each $i, j \in N$, $\exists\ g \in A_i^t$ such that $v_i(A_i^t) \geq v_i(A_j^t \setminus \{g\})$
   - Therefore, for each $i, j \in N$, $\exists\ g \in A_j^{t+1}$ and $\exists\ j' \in N$ such that

$$v_i(A_i^{t+1}) \overset{(1)}{\geq} v_i(A_i^t) \overset{(3)}{\geq} v_i\left(A_{j'}^t \setminus \{g\}\right) \overset{(2)}{=} v_i(A_j^{t+1} \setminus \{g\})$$

Efficiency+ EF1

- Weak Pareto optimality (WPO)
  - Allocation A is weakly Pareto optimal if there is no allocation B such that $v_i(B_i) > v_i(A_i)$ for all $i \in N$.
  - "Can't make everyone happier"

- Pareto optimality (PO)
  - Allocation A is Pareto optimal if there is no allocation B such that $v_i(B_i) \geq v_i(A_i)$ for all agents $i \in N$, and at least one inequality is strict.
  - "Can't make someone happier without making someone else less happy"

- Neither envy cycle elimination algorithm nor round robin is even weak Pareto optimal (Why?)

EF1+PO for goods

- Maximum Nash Welfare (MNW) to the rescue!
  - Essentially, maximize the Nash welfare across all integral allocations

- Theorem [Caragiannis et al. '16]
  - (Almost true) Any allocation in $\text{argmax}_A \prod_{i \in N} v_i(A_i)$ is EF1 + PO.

EF1+PO for goods

- Proof that A maximizing $\prod_i v_i(A_i)$ is EF1 + PO
  - PO is obvious
    - Suppose for contradiction that there is an allocation B such that $V_i(B_i) \geq V_i(A_i)$ for each i and $V_i(B_i) > V_i(A_i)$ for at least one i
    - Then, $\prod_i V_i(B_i) \geq \prod_i V_i(A_i)$, which is a contradiction
  - EF1 requires a bit more work
    - Fix any agents i, j and consider moving good $g \in A_j$ to A_i,
    - Since A is MNW
    - $\Rightarrow V_i(A_i \cup \{g\}) \cdot V_j(A_j \setminus \{g\}) \leq V_i(A_i) \cdot V_j(A_j)$
    - $\Rightarrow (v_i(A_i) + v_i(g)) \cdot (v_j(A_j) - v_j(g)) \leq v_i(A_i) \cdot v_j(A_j)$
    - $\Rightarrow \cdot \frac{v_j(A_j) - v_j(g)}{v_j(A_j)} \leq \frac{v_i(A_i)}{v_i(A_i) + v_i(g)} = \frac{v_i(A_i) + v_i(g) - v_i(g)}{v_i(A_i) + v_i(g)}$
    - $\Rightarrow 1 - \frac{v_j(g)}{v_j(A_j)} \leq 1 - \frac{v_{i(g)}}{v_i(A_i \cup \{g\})} \Rightarrow \frac{v_j(g)}{v_j(A_j)} \geq \frac{v_i(g)}{v_i(A_i \cup \{g\})} \geq \frac{v_i(g)}{v_i(A_i \cup \{g^*\})}$
      - where $g \in A_j$ is the good liked the most by i
    - $\Rightarrow \sum_{g \in A_j} \frac{v_j(g)}{v_j(A_j)} \geq \sum_{g \in A_j} \frac{v_i(g)}{v_i(A_i \cup \{g^*\})}$
    - $\Rightarrow v_i(A_i \cup \{g^*\}) \geq v_i(A_j)$

What is wrong in these arguments?

EF1+PO for goods

- Edge case: all allocations have zero Nash welfare
  - E.g., allocate two goods between three agents
  - Allocating both goods to a single agent can violate EF1
  - Requires a slight modification of the rule in this edge case
    - Step 1: Choose a subset of agents $S \subseteq N$ with largest $|S|$ such that it is possible to give a positive utility to each agent in S simultaneously
    - Step 2: Choose $\text{argmax}_A \prod_{i \in S} V_i(A_i)$
  - Quick questions:
    - How does this fix the example above?

Computation

- For indivisible goods, finding an MNW allocation is strongly NP-hard (NP-hard even if all values are bounded)

- Open Question:
  - Can we compute *some* EF1+PO allocation in polynomial time?
  - [Barman et al., ‘17]:
    - There exists a pseudo-polynomial time algorithm for finding an EF1+PO allocation
      - Time is polynomial in n, m, and $\max_{i,g} v_{i,g}$
      - Already better than the time complexity of computing an MNW allocation

Envy-Freeness up to One Bad (EF1)

- An allocation is envy-free up to one bad (EF1) if, for all agents i, j, there exists a bad $g \in A_i$ for which

$$v_i(A_i \setminus \{g\}) \geq v_i(A_j))$$

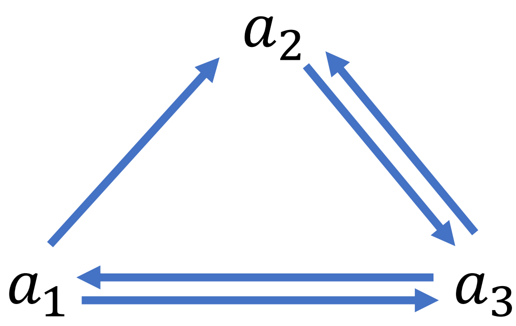
- "Agent i may envy agent j, but the envy can be eliminated by removing a single bad from i's bundle."
- Question:
  - Does round robin satisfy EF1?
  - Does envy cycle elimination satisfy EF1?

Envy Cycle Elimination Algorithm for Bads

- Envy graph: Edge from i to j if i envies j

- Natural Variation of Envy Cycle Elimination
  - One at a time, allocate a good to an agent that envies no one
  - While there is an envy cycle, rotate the bundles along the cycle.

Envy Cycle Elimination Algorithm for Bads



	g_1	g_2	g_3	g_4	g_5	g_6	g_7
a_1	-1	-4	-2	-3	0	-1	-3
a_2	-2	-1	-2	-2	-3	-1	-4
a_3	-1	-3	-1	-1	-3	-10	-2

Envy Cycle Elimination Algorithm for Bads

- Theorem [Bhaskar et al. 2021]: A variation of Envy Cycle Elimination, in which the cycles that are eliminated are carefully chosen, satisfies EF1
  - Each agent has an outgoing edge only to the agent whom she envies and whose bundle has maximum utility

EF1 with Goods and Bads [Aziz et al. 2019]

- An allocation is envy-free up to one item (EF1) if, for all agents i, j, there exists an item $o \in A_i \cup A_j$ for which

$$v_i(A_i \setminus \{o\}) \geq v_i(A_j \setminus \{o\})$$

- Round robin fails EF1

	o_1	o_2	o_3	o_4
a_1	2	1	-4	-4
a_2	2	-3	-4	-4

Double Round Robin

- Let $O^- = \{o \in O: \forall i \in N, v_i(o) \leq 0\}$ denote all unanimous bads and $O^+ = \{o \in O: \exists i \in N, v_i(o) > 0\}$ denote all objects that are a good for some agent.
  - Suppose that $|O^-| = an$ for some $a \in \mathbb{N}$. If not, add dummy bads with $v_i(o) = 0$ for all $i \in N$.

- Double round robin:
  - Phase 1: O^- is allocated by round robin in order $(1, 2, \ldots, n-1, n)$
  - Phase 2: O^+ is allocated by round robin in order $(n, n-1, \ldots, 2, 1)$
  - Agents can choose to skip their turn in phase 2

Double Round Robin

O^- (o_1, o_2) O^+ (o_3, o_4, o_5, o_6)

	o_1	o_2	o_3	o_4	o_5	o_6
a_1	-3	-4	2	4	-4	-4
a_2	-4	-8	10	10	3	1

Double Round Robin

- Theorem [Aziz et al. 2019]: The double round robin algorithm outputs an allocation that is EF1 for combinations of goods and bads in polynomial time
- Proof:
  - Consider $i, j \in N$ with $i < j$
  - i does not envy j for more than one items
    - $u_i(A_i \cap O^-) \geq u_i(A_j \cap O^-)$, since i chooses before j in phase 1
    - $u_i(A_i \cap O^+) \geq u_i(A_j \cap O^+ \setminus \{g\})$ since j chooses at most once before i in phase 2
  - j does not envy i for more than one items
    - $u_j(A_j \cap O^- \setminus \{g\}) \geq u_j(A_i \cap O^-)$, since i chooses just once before j in phase 1
    - $u_j(A_j \cap O^+) \geq u_j(A_i \cap O^+)$ since j chooses before i in phase 2

EF1 + PO for Bads

- Theorem [Aziz et al. 2019]:
  - When items can be either goods or bads and $n = 2$, an EF1 + PO allocation always exists and can be found in polynomial time

- Theorem [Ebadian et al. 2022; Garg et al. 2022]:
  - For bivalued instances, an EF1 + PO allocation always exists and can be found in polynomial time

Open Problem:
Does an EF1 + PO allocation always exist for bads?

EFX

- Envy-freeness up to any good (EFX)
  - $\forall i, j \in N, \forall g \in A_j : V_i(A_i) \geq V_i(A_j \backslash \{g\})$
  - In words, i shouldn't envy j if she removes *any* good from j's bundle
  - EFX ⇒ EF1 $\left(\forall i, j \in N, \exists g \in A_j : V_i(A_i) \geq V_i(A_j \backslash \{g\})\right)$
- EF1 vs EFX example:
  - {A → P1; B,C → P2} is EF1 but not EFX, whereas .
  - {A,B → P1; C → P2} is EFX.

	A	B	C
P1	5	1	10
P2	0	1	10

- Open question: Does there always exist EFX allocation?

EFX

- (Easy to prove) EFX allocation always exists when…
  - Agents have identical valuations (i.e. $V_i = V_j$ for all i, j)
  - Agents have binary valuations (i.e. $v_{i,g} \in \{0,1\}$ for all i, g)
  - There are $n = 2$ agents with general additive valuations

- But answering this question in general (or even in some other special cases) has proved to be surprisingly difficult!

EFX: Recent Progress

- Partial allocations
  - [Caragiannis et al., '19]: There exists a partial EFX allocation A that has at least half of the optimal Nash welfare
  - [Ray Chaudhury et al., '19]: There exists a partial EFX allocation A such that for the set of unallocated goods U, $|U| \leq n - 1$ and $V_i(A_i) \geq V_i(U)$ for all i
- Restricted number of agents
  - [Ray Chaudhury et al., '20]: There exists a complete EFX allocation with $n = 3$ agents
- Restricted valuations
  - [Amanatidis et al., '20]: Maximizing Nash welfare achieves EFX when there exist a, b such that $v_{i,g} \in \{a, b\}$ for all i, g

(Relaxed) Equitability

Equitability

- Recall equitability:

$$\forall i, j \in N: v_i(A_i) \geq v_j(A_j)$$

- We can relax it in the same way we did for envy-freeness [Gourves et al. 2014, Freeman et al. 2019].
- Equitability up to one good (EQ1):

$$\forall i, j \in N, \exists g \in A_j: v_i(A_i) \geq v_j(A_j \setminus \{g\})$$

- Equitability up to any good (EQX):

$$\forall i, j \in N, \forall g \in A_j: v_i(A_i) \geq v_j(A_j \setminus \{g\})$$

Algorithm for Achieving EQX

- Greedy Algorithm [Gourves et al. 2014]:
  - Allocate to the lowest-utility agent the unallocated good that she values the most.
- Almost the same as EF1 algorithm, but achieves EQX!
  - Compare to EFX, existence still unknown

EQ1/EQX + PO

- Theorem [Freeman et al. 2019]:
  - An allocation satisfying EQ1 and PO may not exist.
  - Compare to EF1 + PO always exists

	g_1	g_2	g_3	g_4	g_5	g_6
a_1	1	1	1	0	0	0
a_2	0	0	0	1	1	1
a_3	0	0	0	1	1	1

In Summary

- Round Robin and Envy Cycle Elimination Algorithm satisfy EF1 but not weak PO for goods
- MNW achieves EF1+PO for goods
- Round Robin satisfies EF1 for bads
- A careful implementation of Envy Cycle Elimination Algorithm satisfies EF1 for bads
- Round Robin does not satisfy EF1 for mixed items, but Double Round Robin does
- EF1+PO allocation for bads is a major open question
- EFX allocation is a major open question
- EQX allocation always exists
- EQX+PO allocation does not always exist

Real Life Applications

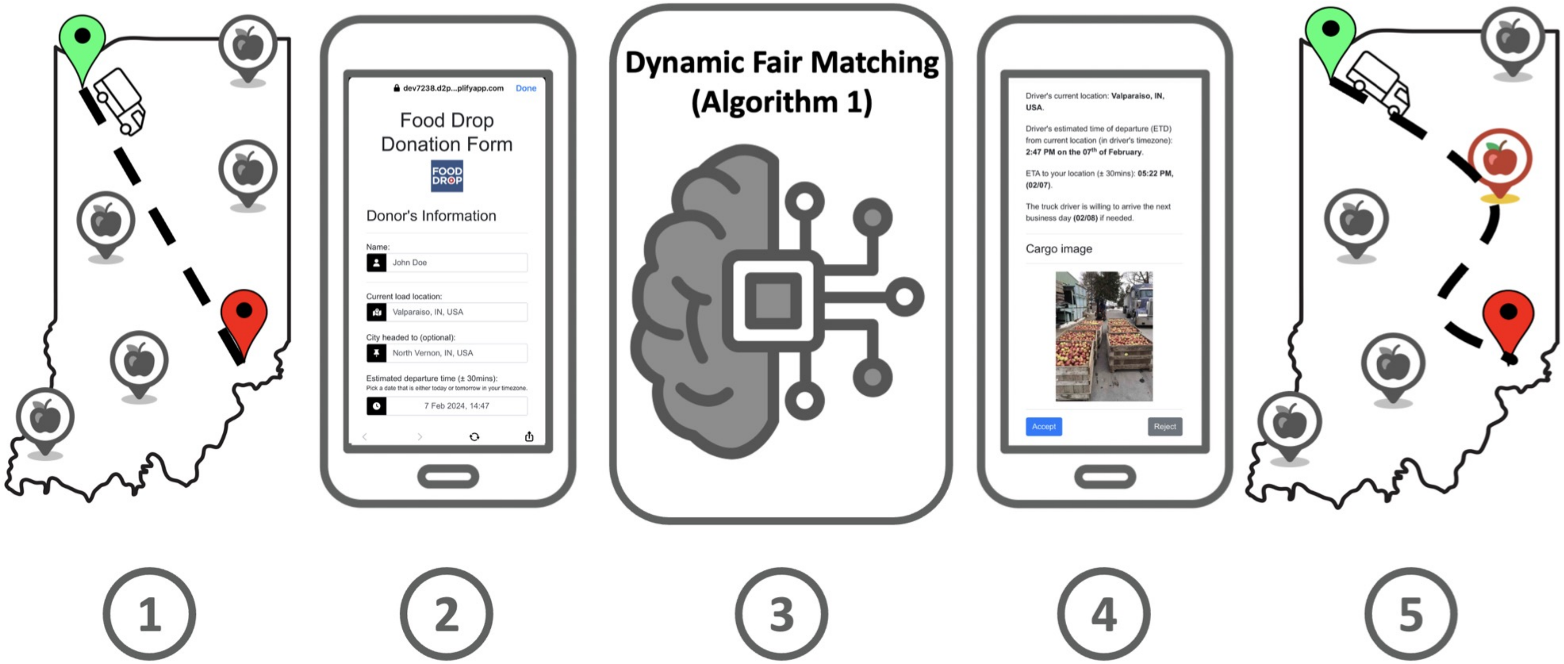
Food Bank [Mertzanidis et al. 2024]

- About a third of the world’s food is thrown away
- In the United States, between 30 and 40 percent of food is wasted, which roughly equals 149 billion meals
- In 2021, close to 12 percent of the global population, or, equivalently, 928 million people, were food insecure
- In the United States, 44.2 million people lived in food-insecure households in 2022
- Food rescue organizations worldwide are leading programs aimed at addressing food waste and food insecurity

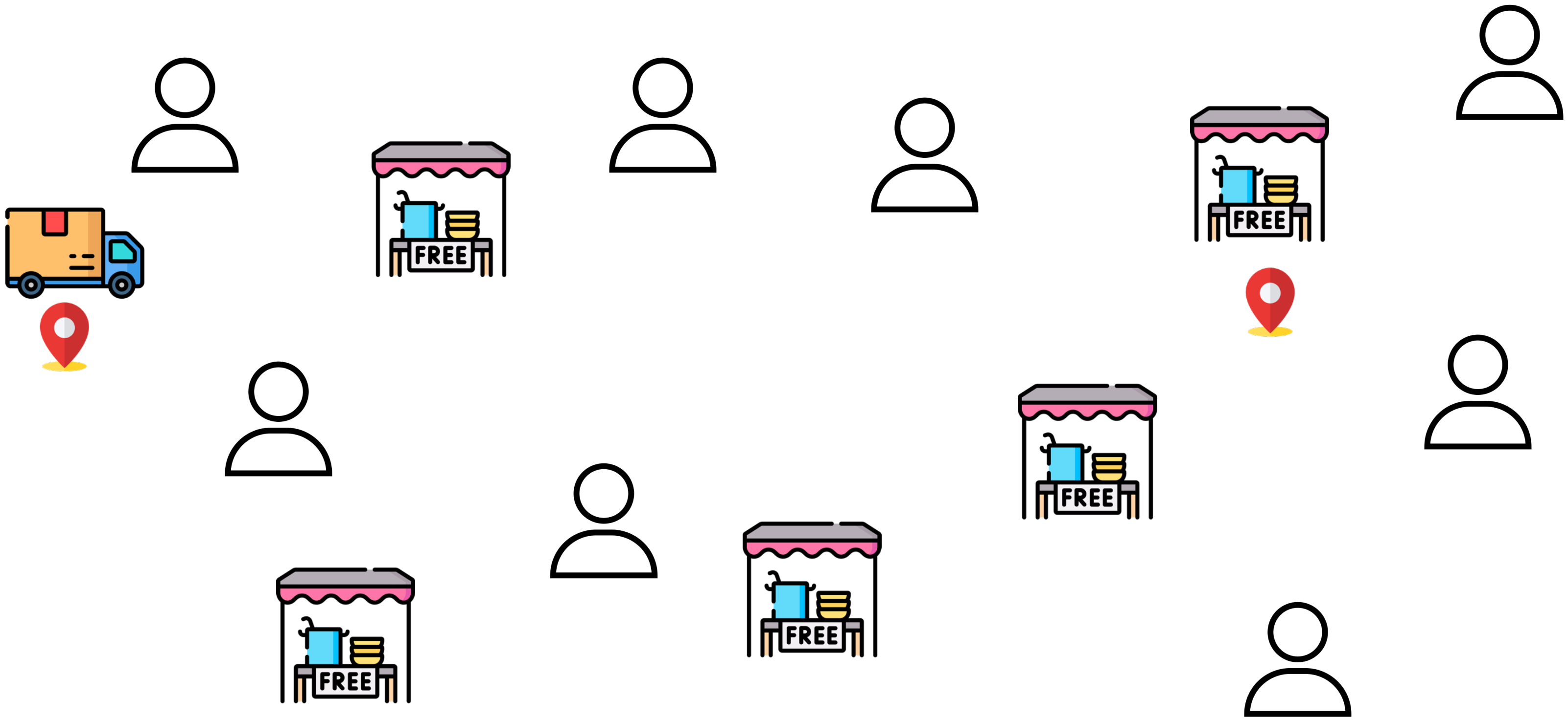
Food Bank [Mertzanidis et al. 2024]

- Food Drop is such a program, in the state of Indiana.
- Food Drop matches truck drivers with rejected truckloads of food
- The average amount of food matched per month is 10,447 lbs
- Matching decisions were manually made
  - check availability and willingness to accept each donation from the food bank's side
  - facilitate the exchange of contact information between the food bank and truck driver,
  - and so on.

Food Bank [Mertzanidis et al. 2024]



Food Bank [Mertzanidis et al. 2024]



- Efficiency for drivers
- Envy-freeness for individuals

Food Bank [Mertzanidis et al. 2024]