Fair and Efficient Social Decision-Making

CSCI 699

Fair Division 3: Best-of-Both Worlds

Evi Micha

Indivisible Goods

- Agents: $N = \{1, 2, \ldots, n\}$
- Resource: Set of indivisible goods $M = \{g_1, g_2, \ldots, g_m\}$
- Allocation $A = (A_1, \ldots, A_n)$is a partition of M
- Each agent i has a valuation $v_i : 2^M \rightarrow \mathbb{R}_+$
  - Additive utilities: $v_i(X) = \sum_{g \in X} v_i(g)$

Lottery

- Envy-freeness is not guaranteed
- *But what about envy − freeness in expectation?*
- Lottery: Distribution over Allocations

Lottery to Random Allocation

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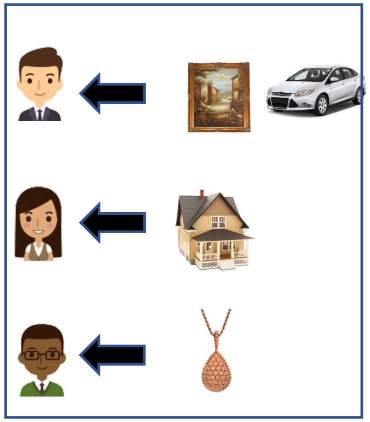

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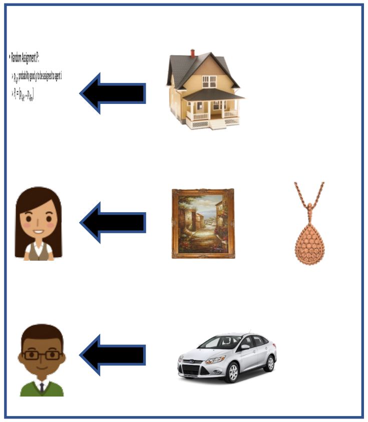

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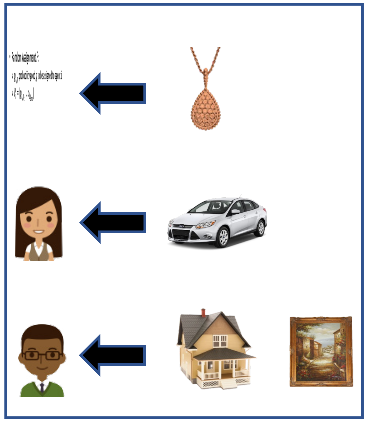

				
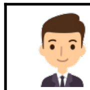	1/4	1/4	1/4	1/2
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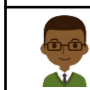	1/2	1/4	1/2	1/4

- Random Assignment P:
  - $p_{i,g}$: probability good g to be assigned to agent i
  - $P_i = [p_{i,g_1}, \dots p_{i,g_m}]$

Envy-Free Random Allocation

- Expected Utility of agent i over a random allocation P
  - $E[v_i(P_i))] = \sum_{g \in M} p_{i,g} \cdot u_i(g)$
- Ex-Ante Envy-Freeness: $\forall i, j \in N,\ E[v_i(P_i)] \geq E[v_i(P_j)]$

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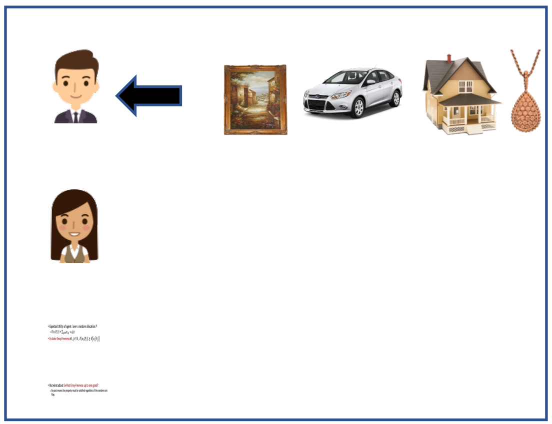

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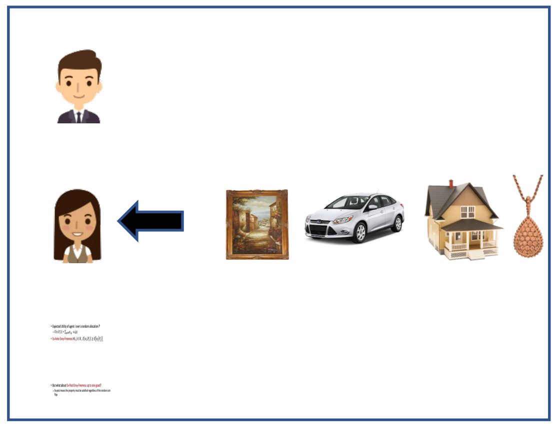

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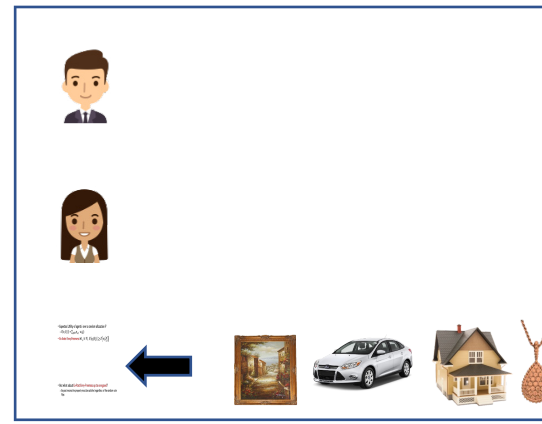


- But what about Ex-Post Envy-Freeness up to one good?
  - Ex-post means the property must be satisfied regardless of the random coin flips

Randomized Round-Robin

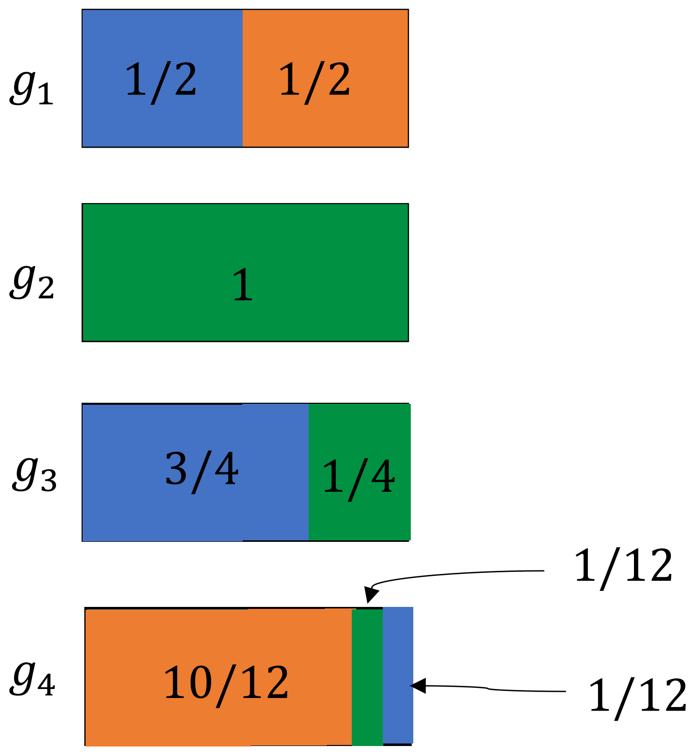
- Round-Robin algorithm ensures EF1
- A natural approach is to choose the agent ordering uniformly at random
- Theorem [Freeman et al., 2020]: Randomized Round-Robin violates ex-ante envy-freeness

Probabilistic Serial Algorithm

- This rule defines a random allocation
- Pretend that each good is a divisible good
- At every point time, all agents “eat” their favorite available good at the same rate
- When all the items are eaten, each agent has eaten m/n “probability shares”

Probabilistic Serial Algorithm

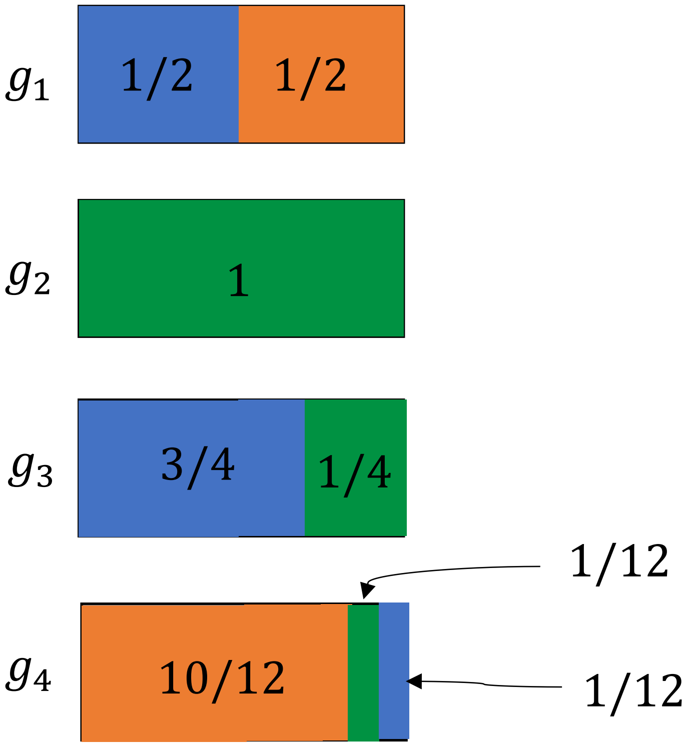
a	***b***	***c***
g_1	g_1	g_2
g_3	g_4	g_1
g_2	g_2	g_3
g_4	g_3	g_2



Probabilistic Serial Algorithm

- Random Allocation

	a	b	c
g_1	$1/2$	$1/2$	0
g_2	0	0	1
g_3	$3/4$	0	$1/4$
g_4	$1/12$	$10/12$	$1/12$



Probabilistic Serial Algorithm

- Theorem [Bogomolnaia and Moulin, 2001]: Probabilistic Serial (PS) Algorithm satisfies ex-ante envy-freeness
- Proof:
  - Whenever an agent i is eating a good g, this good g is the best available good for agent i at that time
  - Therefore, agent i weakly prefers good g over any other good that another agent i is eating at the same time
  - Since all the agents consume at the same rate, the random assignment is ex-ante envy-free

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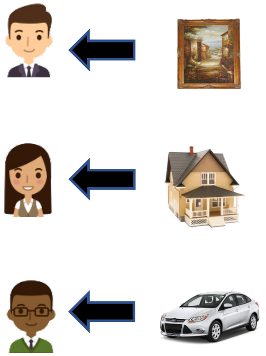
Random Allocation to Lottery

- We saw that every lottery induces a random assignment. Is the converse also true?
- A permutation matrix is a square binary matrix that has exactly one entry of 1 in each row and each column
- When $n = m$, a permutation matrix represents an allocation

			
	1	0	0
	0	0	1
	0	1	0



Random Allocation to Lottery

- A bistochastic matrix is a square non-negative matrix, each of whose rows and columns sums to 1
- When $n = m$, a bistochastic matrix represents a random allocation

			
	1/2	1/4	1/4
	1/2	1/2	0
	0	1/4	3/4

Random Allocation to Lottery

	1/2	1/4	1/4
	1/2	1/2	0
	0	1/4	3/4

- Theorem [Birkhoff-von Neumann]: Any $k \times k$ bistochastic matrix can be obtained as a convex combination of at most $O(k^2)$ permutation matrices

	0	0	1
	1	0	0
	0	1	0

$\times \mathbf{1/4}$

	0	1	0
	1	0	0
	0	0	1

$\times \mathbf{1/4}$

	1	0	0
	0	1	0
	0	0	1

$\times \mathbf{1/2}$

PS-Lottery Algorithm

- When $n = m$, PS algorithm returns a bistochastic matrix
- We want to apply PS algorithm in a way that returns a bistochastic matrix even when $n \neq m$
- Add dummy items such that $m' = n \cdot c$
- For each agent i, create c representatives, $\{i_1, \dots, i_c\}$
- Each representative i_t eats their favorite available good during time step $[t-1, t]$

PS-Lottery Algorithm

a	***b***	***c***
g_1	g_1	g_2
g_3	g_4	g_1
g_2	g_2	g_3
g_4	g_3	g_2

PS-Lottery Algorithm

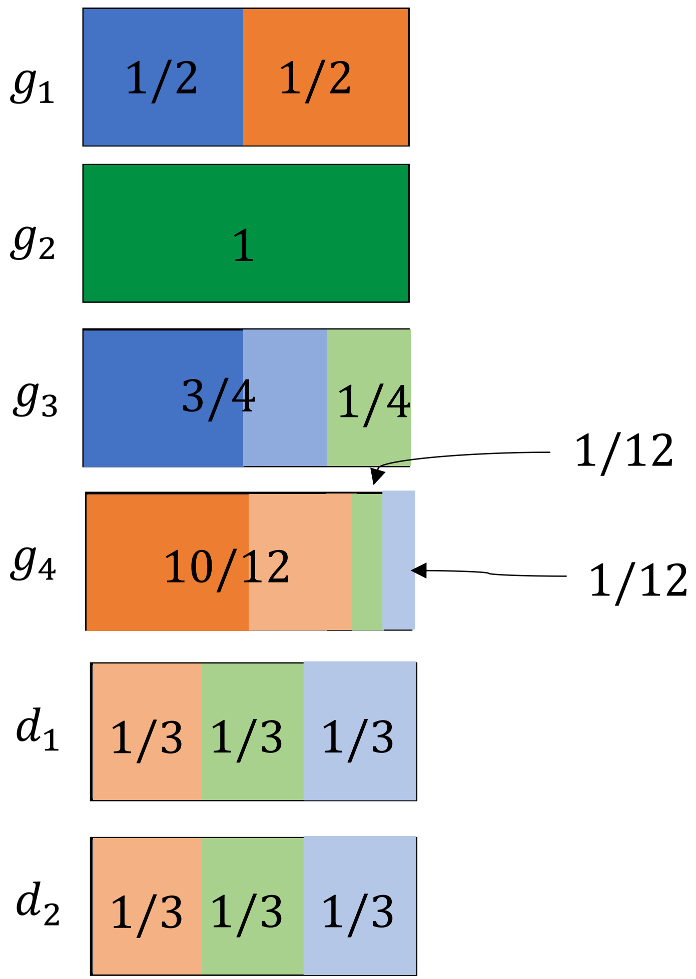
a	b	c
g_1	g_1	g_2
g_3	g_4	g_1
g_2	g_2	g_3
g_4	g_3	g_2
d_1	d_1	d_1
d_2	d_2	d_2

PS-Lottery Algorithm

a_1	a_2	b_1	b_2	c_1	c_2
g_1	g_1	g_1	g_1	g_2	g_2
g_3	g_3	g_4	g_4	g_1	g_1
g_2	g_2	g_2	g_2	g_3	g_3
g_4	g_4	g_3	g_3	g_2	g_2
d_1	d_1	d_1	d_1	d_1	d_1
d_2	d_2	d_2	d_2	d_2	d_2

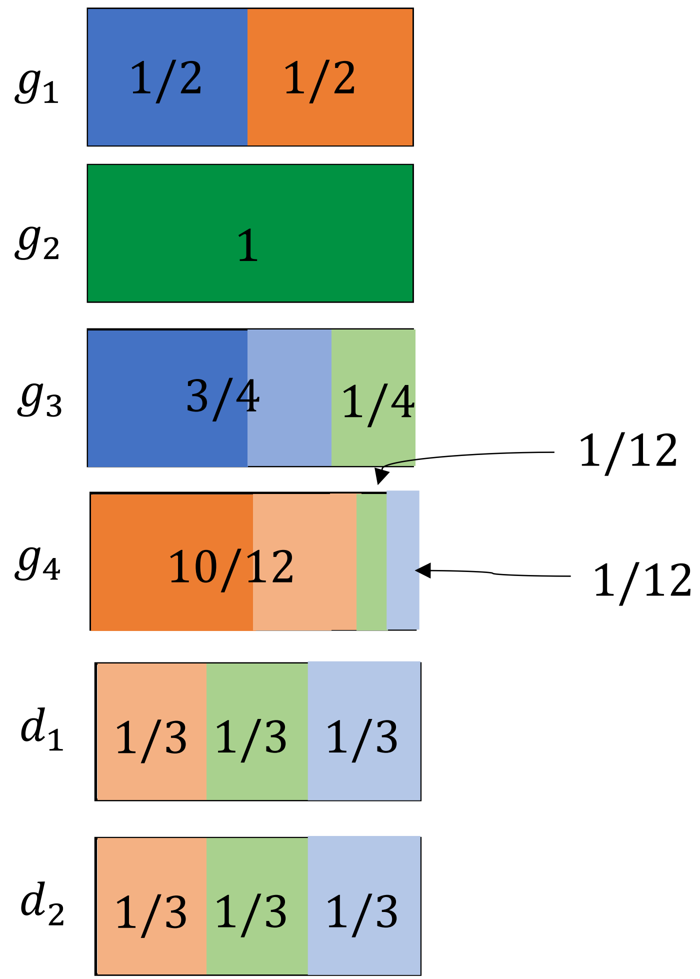
PS-Lottery Algorithm

a_1	a_2	b_1	b_2	c_1	c_2
g_1	g_1	g_1	g_1	g_2	g_2
g_3	g_3	g_4	g_4	g_1	g_1
g_2	g_2	g_2	g_2	g_3	g_3
g_4	g_4	g_3	g_3	g_2	g_2
d_1	d_1	d_1	d_1	d_1	d_1
d_2	d_2	d_2	d_2	d_2	d_2



PS-Lottery Algorithm

	a_1	a_2	b_1	b_2	c_1	c_2
g_1	1/2	0	1/2	0	0	0
g_2	0	0	0	0	1	0
g_3	1/2	1/4	0	0	0	1/4
g_4	0	1/12	1/2	1/3	0	1/12
d_1	0	1/3	0	1/3	0	1/3
d_2	0	1/3	0	1/3	0	1/3



PS-Lottery Algorithm

	a_1	a_2	b_1	b_2	c_1	c_2
g_1	1/2	0	1/2	0	0	0
g_2	0	0	0	0	1	0
g_3	1/2	1/4	0	0	0	1/4
g_4	0	1/12	1/2	1/3	0	1/12
d_1	0	1/3	0	1/3	0	1/3
d_2	0	1/3	0	1/3	0	1/3

- Apply Birkoffs' decomposition for getting a lottery
- In each allocation A in the support of the lottery, each i_t is assigned exactly one item, denoted by g_i^t, that she ate during the time step $[t-1, t]$
- i is assigned $A_i = \{g_i^1, \ldots, g_i^t\}$

PS-Lottery Algorithm

- Theorem [Aziz et al., 2023]: PS-Lottery Algorithm satisfies ex-post EF1
- Proof:
  - Consider any two representatives i_t and $j_{t'}$ with $t' > t$
  - Since $j_{t'}$ is assigned $g_j^{t'}$ this means that $g_j^{t'}$ was available during the time step $[t' - 1, t']$, hence during the time step $[t - 1, t]$
  - i_t chose to eat g_i^t instead of $g_j^{t'}$, thus i prefers g_i^t to $g_j^{t'}$
  - Consider any two agents i and j
    - From above, $\forall t \in [C], u_i(g_i^t) \geq u_i(g_i^{t+1})$
    - Thus, $u_i(A_i) = u_i(g_i^1) + u_i(g_i^2) + \cdots + u_i(g_i^t) \geq u_i(g_j^2) + u_i(g_j^2) + \cdots + u_i(g_j^t) = u_i(A_j \setminus g_j^1)$

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