Fair and Efficient Social Decision Making (CSCI 699)
Assignment 1
Due Date: October 18th, 11am

Notes

- Citation policy
  - It is certainly preferable for you to solve the questions without consulting a peer or a published source. However, if you do consult and obtain useful insights, you must cite the name of the peer or the link of the source you referred.
  - Further, you should write the solution in your own words. The best way to do so is to not take any notes during discussions, spend at least a few hours doing something else, and then construct the solution on your own.
- No garbage policy: Stating 'I do not know how to approach this' gets you 20% of the points (this also applies to a subproblem).
- Typed assignments are highly preferred (LaTeX or Word), especially if your handwriting is possibly illegible or if you do not have access to a good quality scanner. Please submit a single PDF.

Question 1 [10 Points] Voting Rules

An election was held in a class of 100 students, in which they provided ranked votes over 4 candidates: Alex, Charlie, Pat, and Kim. The votes are as follows:

- 32 students voted Alex $\succ$ Kim $\succ$ Charlie $\succ$ Pat
- 27 students voted Charlie $\succ$ Pat $\succ$ Kim $\succ$ Alex
- 25 students voted Pat $\succ$ Charlie $\succ$ Kim $\succ$ Alex
- 16 students voted Kim $\succ$ Pat $\succ$ Alex $\succ$ Charlie

Determine the winner of the election according to plurality, Borda, STV, and Kemeny.
[Hint: Checking whether the profile admits a Condorcet winner can help simplify your calculations.]

Question 2 [25 Points]Gibbard-Satterthwaite Theorem

Recall four definitions we introduced in class when proving the Gibbard - Satterthwaite theorem.

Strategyproofness: A voting rule f is strategyproof if for every preference profile $\overrightarrow{\succ}$ and every voter i, there does not exist another preference profile $\overrightarrow{\succ}'$ with $\succ'_j = \succ_j$ for all $j \neq i$ such that $f(\overrightarrow{\succ}') \succ_i f(\overrightarrow{\succ})$.

Ontoness: A voting rule f is onto if for every alternative a, there exists a preference profile $\vec{\succ}$ such that $f(\vec{\succ}) = a$.

Strong monotonicity: A voting rule f is strongly monotonic if for every preference profile $\vec{\succ}$ and preference profile $\vec{\succ}'$ such that

$$\forall i, a : f(\vec{\succ}) \succ_i a \Rightarrow f(\vec{\succ}) \succ'_i a,$$

then it must be the case that $f(\vec{\succ}') = f(\vec{\succ})$.

Pareto optimality: A voting rule f is Pareto optimal if $f(\vec{\succ}) \neq a$, whenever there exists b such that $b \succ_i a$ for every voter i.

In class, we remarked the following facts without proof and used them to derive the Gibbard- Satterthwaite theorem. In this question, your goal is to prove them (of course, without using the Gibbard-Satterthwaite theorem itself).

(a) [15 Points] Prove that every strategyproof voting rule is strongly monotonic.
[Hint: To prove the contrapositive, suppose a pair of profiles $\vec{\succ}$, $\vec{\succ}'$ demonstrate a violation of strong monotonicity. Move from $\vec{\succ}$ to $\vec{\succ}'$ "one vote at a time", and find a violation of strategyproofness somewhere along the way.]

(b) [10 Points] Prove that every strategyproof and onto voting rule is Pareto optimal.
[Hint: You can use part (a) to prove this.]

Question 3 [25 Points] Strategyproof Voting Rules

Consider the following randomized voting rule we covered in class, which achieves $O(\sqrt{m \log(m)})$ distortion.

1. Given the preference profile $\vec{\succ}$, define $sc(a, \vec{\succ}) = \sum_i 1/r(a, \succ_i)$ where $r(a, \succ_i)$ is the rank of alternative a in the preference ranking $\succ_i$ of voter i.
2. With probability 1/2, execute the rule which selects each alternative a with probability proportional to $sc(a, \vec{\succ})$, i.e. with probability $\frac{sc(a,\vec{\succ})}{\sum_b sc(b,\vec{\succ})}$.
3. With probability 1/2, execute the rule which selects an alternative uniformly at random, i.e., each alternative a with probability $1/m$.

(a) [10 Points] Show that this voting rule is a mixture over unilaterals and duples. You need to produce an explicit probability distribution over unilaterals and duples which, on every input profile, selects each alternative with the same probability as the above rule does.

(b) [15 Points] Recall that being a mixture over unilaterals and duples is a necessary but not a sufficient condition for being strategyproof. Nonetheless, show that this voting rule is strategyproof. That is, prove that a voter can never increase her expected utility by reporting a ranking different from an honest ranking consistent with their utility function.

Question 4 [15 Points] Distortion

In class, we learned that among all deterministic voting rules with access to the preference profile, the plurality rule achieves asymptotically optimal distortion $\Theta(m^2)$, despite the fact that it accesses only a part

of the preference profile — just the plurality votes. In other words, information contained in the preference profile other than the plurality votes is not significantly useful.
What happens with randomized voting rules?

(a) [5 Points] Show that there is a randomized voting rule which achieves $O(m)$ distortion without using any preference information whatsoever (that is, this voting rule is a constant function).

(b) [10 Points] Show that every randomized voting rule which has access to only the plurality votes incurs $\Omega(m)$ distortion. This suggests that for randomized rules, plurality votes are almost useless on their own, and we must use the additional information contained in the preference profile to achieve lower distortion.

Question 5 [25 Points] Committee Selection

Recall the EJR guarantee for approval-based committee selection we discussed in class. A committee W satisfies EJR if

- for all $S \subseteq N$, $\ell \in \{1, \dots, k\}$, with $|S| \geq \ell \cdot n/k$ and $|\cap_{i \in S} A_i| \geq \ell$
- $u_i(W) \geq \ell$ for some $i \in S$.

Now, consider the following guarantee for approval-based committee selection which is called EJR+. A committee W satisfies EJR+ if

- for all $S \subseteq N$, $\ell \in \{1, \dots, k\}$, $c \in M \setminus W$ with $|S| \geq \ell \cdot n/k$ and $c \in \cap_{i \in S} A_i$
- $u_i(W) \geq \ell$ for some $i \in S$.

Lastly recall the FJR guarantee we discussed in class. A committee W satisfies FJR if

- for all $S \subseteq N$, $T \subseteq M$ and $\beta \in \{1, \dots, k\}$, with $|S| \geq |T| \cdot n/k$ and $u_i(T) \geq \beta$
- $u_i(W) \geq \beta$ for some $i \in S$.

(a) [10 Points] Prove that PAV satisfies EJR+.
[Hint: Revisit the proof we covered in class showing that PAV satisfies EJR.]

(b) [10 Points] Show that EJR+ implies EJR, but EJR does not imply EJR+.

(c) [5 Points] Prove that EJR+ and FJR are incomparable.
[Hint: You can use (a) for showing that EJR+ does not imply FJR. For showing that FJR does not imply EJR+, consider a profile with two voters, three candidates and $k = 2$.]